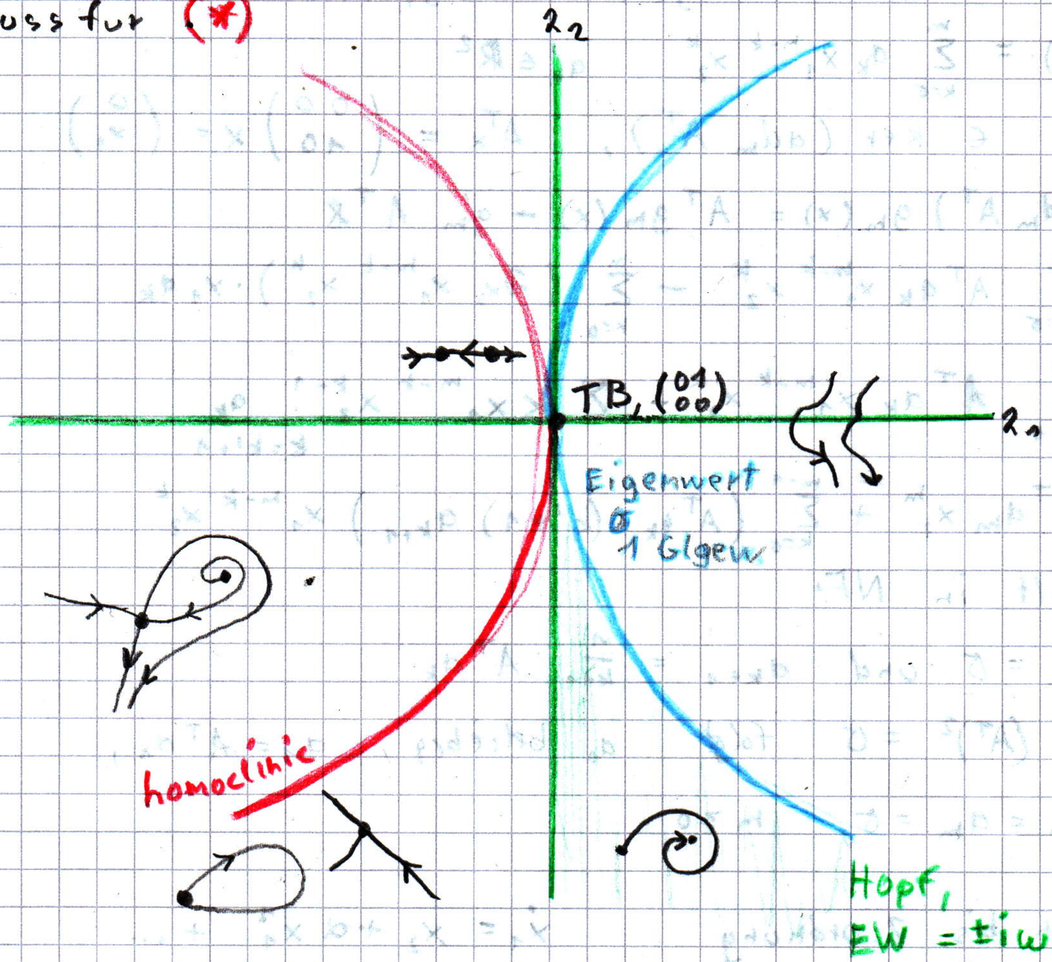
$\varphi_3 = \Xi(\mu, x)$ stetig, $\Xi(\mu, \cdot)$ C^0-Homöo

Fluss für (*)



3

6.5 Iterations

26.04.2011

Consider $F = Bx + \ldots \in C_1^\infty(B)$ with $B \in GL(N)$ invertible, i.e. a local diffeo F with fixed point $0 \in \mathbb{R}^N$

Trf $x = \Phi(y) = y + \ldots \in C_1^\infty(\mathrm{id})$, loc. Diffeo transforms iteration $x_{n+1} = F(x_n)$ to $y_{n+1} = G(y_n)$ $= (\Phi^{-1} \circ F \circ \Phi)(y_n) = By_n + G_2(y_n) + \ldots$

What does this do to the coefficients?

$(\Phi \circ G)(y) = (F \circ \Phi)(y)$ Compare order m in y:

$$G_m + \ldots + \Phi_m \circ B = F_m + \ldots + B \circ \Phi_m$$

Where "..." are terms involving $G_2, \ldots, G_{m-1}, B, F_2, \ldots, F_{m-1}, \Phi_2, \ldots \Phi_{m-1}$. Therefore

$$G_m = B \circ \Phi_m - \Phi_m \circ B + R_m(\ldots)$$ p.o.t.

Therefore we can choose $\Phi \in C_1^\infty(\mathrm{id})$ such that

$G_m \in (\text{Range } B_m[\cdot])^\perp = \text{Kern}(B_m[\cdot])$ where
$B_m[\phi_m] := B \circ \Phi_m - \Phi_m \circ B$ is linear and $\langle \cdot , \cdot \rangle$
is scalar product on H_m

Theorem: For any $F \in C_1^\infty(B)$, $B \in GL(n)$, there exists
$\Phi \in C_1^\infty(id)$ such that $G := \Phi^{-1} \circ F \circ \Phi$ is in
$B^T[\cdot]$ normal form i.e.:

$$B^T G_m = G_m \circ B^T \qquad (\forall m \geq 2)$$

Therefore all these G_m commute with any
$\gamma \in T := \text{clos}\{ (B^T)^n \mid n \in \mathbb{Z} \}$.
If $BB^T = B^T B$ is normal, then $\gamma \cdot (T_m G) = (T_m G) \circ \gamma$
$\big((T_m G)(y) := By + \dots + G_m(y)\big)$ $(\forall \gamma \in \Gamma, m \geq 2)$

Proof: Show $(B_m[\cdot])^T \overset{!}{=} B_m^T[\cdot]$, with some sc. prod
$\langle \cdot , \cdot \rangle$ on H_m as in section **6.3**

Claim: $\langle B_m^T[f], g \rangle \overset{!}{=} \langle f, B_m[g] \rangle \qquad (\forall f, g \in H_m)$

$$\overset{\|}{\langle B^T f, g\rangle - \langle f \circ B^T, g \rangle} \qquad \langle f, Bg \rangle - \langle f, g \circ B \rangle$$

trivial by sc.-product in $\mathbb{R}^n$, step in **6.3**

$\square$

Example 1: Period doubling

Consider $\mathbb{R}^1$, $F(x) = -x + F_2 + \dots$
$B = -1 = B^T$, $\Gamma = \mathbb{Z}_2 = \langle -1 \rangle$
Normal form: $G(-x) = -G(x)$, odd
(to any finite order, omit ...) Hence we have
to study

$$y_{n+1} = G(y_n) = y_n g(y_n^2)$$

This amounts to an iteration of

$$|y_{n+1}| = |y_n| \cdot |g(y_n^2)|$$

In particular any orbit $|y_n|$ which stays near $y=0$ must converge to a fixed point $|y_*|$,

$$g(y_*^2)=1.$$

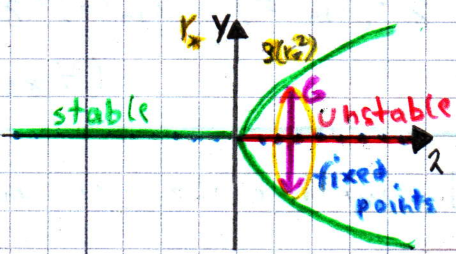
In the parameter dependent case $F(\lambda,x)$ with $F(0,x) = -x+\dots$ we can proceed analogously to get

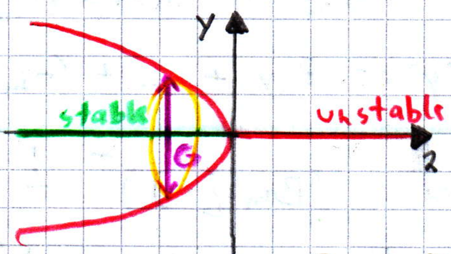
$$y_{n+1} = G(\lambda, y_n) = y_n \cdot g(\lambda, y_n^2)$$

To solve $g(\lambda, y_*^2)=1$ near $g(0,0)=1$ we involve implicit function theorem:

For $\partial_\lambda g(\lambda=0, y_*^2=0) \neq 0$ we get solution branch $\lambda = \lambda(y_*^2)$ such that $G(\lambda, \{\pm y_*\}) = \mp y_*$

(use $G(0,y) = -y+\dots$)





$\partial_2 g<0$ $\qquad\qquad$ $\partial_2 g>0$

$(\partial_\lambda g>0)$ $\qquad\qquad$ $(\partial_\lambda g>0)$

direct: study $F\circ F$, seek fixed points

Example 2: Bifurcation of invariant "circles"

[Normark, Sacker-Sell (1960s)]

$$F(x) = Bx+\dots \in \mathbb{R}^2, \qquad B = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

rotation by ω, $\operatorname{spec} B = \pm e^{i\omega}$. Assume $\frac{\omega}{2\pi} \notin \mathbb{Q}$ irrational. Then $\Gamma = S^1 = SO(2)$

Normal form G commutes with rotations. Hence complex rotation $\underbrace{y_1 + i y_2}_{\mathbb{R}^2} = re^{i\varphi} = z$ produces iteration (with parameter)

$\boxed{z^{\ell+1}\bar{z}^{\ell}}$

$$z_{n+1} = G(\lambda, z_n, \bar z_n) = z_n \cdot g(\lambda, |z_n|^2), \quad g(0,0)=e^{i\omega}$$

i.e. $\quad r_{n+1} = r_n \cdot h^R(\lambda, r_n^2) \qquad$ (no φ here!)

$\quad \varphi_{n+1} = \varphi_n + h^I(\lambda, r_n^2) \qquad$ where $h^R = |g(\lambda, r_n^2)|$, $h^I = \arg(g(\lambda, r_n^2))$

$\Rightarrow g = h^R e^{ih^I}$.

If solution stays near $r = 0$, then $r_n \to r_*$ with

$h^R(\lambda, r_*^2) = 1$ (see period doubling analisys)

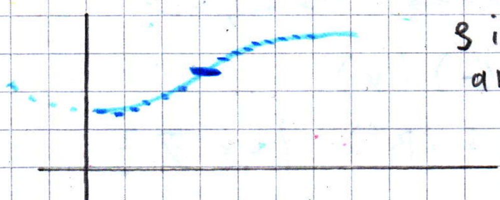
$h^R(0,0) = 1$ e.g. $\lambda = \lambda(r_*^2)$

On these fixed values of r_* we have an iteration

$$\varphi_{n+1} = \varphi_n + \underbrace{h^I(\lambda(r_*^2), r_*^2)}_{=: \omega + r_*^2 \tilde{h}(r_*^2)}$$

This iteration on the invariant circle $r = r_*$ has rotation number $\rho = \dfrac{\omega + r_*^2 \tilde{h}(r_*^2)}{2\pi}$

but: devil's staircase!

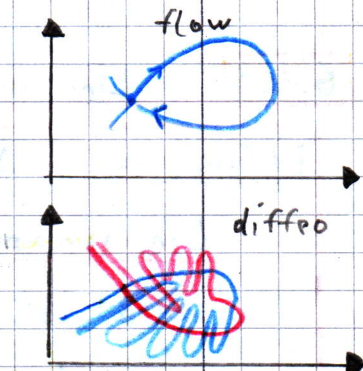


ρ is supposed to be locally constant at any rational value of ρ

plateaus should be very very small

never visible in finite order normal forms

$\dot{x} = Ax$, $x_{n+1} = e^A x_n$ for $x_n := x(n)$



6.6 Normal forms: flows, diffeos, Lie algebras and Lie groups

Theorem [~Takens) (1970s)]

Let $B = e^A$, $AA^T = A^T A$ in $\mathbb{R}^N$. Consider local diffeo Then F can be written as the time = 1 map of the flow of a v.field $f(x) = Ax + \dots$, and the $B^T[\cdot]$ normal form G of F is the time = 1 map of the ad A^T normal form g of f, up to any finite order $m \geq 2$

Schematically:

$$\begin{array}{ccc} F & \xrightarrow{\text{normal form}} & G \\ \text{exp, flow time=1} \updownarrow & /// & \updownarrow \text{flow time=1, exp} \\ f & \xrightarrow[\text{Form}]{\text{Normal}} & g \end{array}$$

More precisely:

Let $F(x) = Bx + \ldots \in C_1^\infty(B)$,

let $\mathcal{F}(f,t)$ denote the flow of $\dot{x} = f(x)$ at time t.

Then (i) there exists a v. field $f \in C_1^\infty(A)$, $f(x) = Ax + \ldots$

such that $T_m F = T_m \mathcal{F}(f, t=1)$ $(\forall m \geq 2)$

(ii) The $ad_m A$ normal form $g(x) = Ax + \ldots = \Phi^* f$

of f satisfies $g_m \in \ker(ad_m(A^T))$ $(\forall m \geq 2)$

for suitable $\Phi \in C_1^\infty(id)$

(iii) let $G := \Phi^{-1} \circ F \circ \Phi$. Then

$T_m G = T_m \mathcal{F}(g, t=1)$ $\forall m \geq 2$

(iv) ⊛ G is in $B^T[\cdot]$ normal form, i.e. ⊗

$(T_m G)(\gamma y) = \gamma (T_m G)(y)$

for all $\gamma \in \Gamma = \text{clos}\{(B^T)^n \mid n \in \mathbb{Z}\}$,

$y \in \mathbb{R}^N$, $m \geq 2$

⊛ Now assume $AA^T = A^T A$. Then

6.6.1 Lie groups & algebras [Bredon][Brocker & Tom Dieck]

Definition: A group $\mathcal{G}$ is called finite-dimensional **Lie group** if it is a finite-dimensional manifold and the composition $F \circ G$ of $F, G \in \mathcal{G}$ is analytic w.r.t. to F, G, in a neighborhood of $e = id \in \mathcal{G}$ and in suitable (homeo) coordinates.

Standard examples: $SO(N)$, $O(N)$, $SE(N)$ $(= SO(N) + \text{translations})$

$SU(N)$, $U(N)$, $SL(N)$, $GL(N)$

$Sp(2N)$, Minkowski, Poincaré, ...

and some non-metric groups.

$S^T \partial S = \partial$, $\partial = \begin{pmatrix} 0 & -id_N \\ id_N & 0 \end{pmatrix}$

Dynamics example: Let *Diff* be the smooth diffeos

$F = Bx + \ldots$, $B \in GL(N)$, with

composition $F_m \circ G := T_m(F \circ G)$

Definition: The Lie algebra $\mathfrak{g}$ of a Lie group $\mathcal{G}$ is the tangent space at $e = \mathrm{id} \in \mathcal{G}$, i.e. $\mathfrak{g} = T_e \mathcal{G}$.
In other words $f \in \mathfrak{g}$ are the tangents $f = \frac{\partial}{\partial t}\Big|_{t=0} F(t)$ of smooth paths $t \to F(t) \in \mathcal{G}$ at $F(0) = e$

Note: Fix $H_0 \in \mathcal{G}$ and consider $F \mapsto F \circ H_0$
via $\frac{\partial}{\partial t}\Big|_{t=0}$ this defines a map $\mathfrak{g} = T_e\mathcal{G} \ni f \mapsto f \circ H_0 \in T_{H_0}\mathcal{G}$

Brief excursion: commutators and Lie-bracket.

For curves $F(t), G(t)$ through e at $t=0$ and $f := \dot{F}(0)$, $g := \dot{G}(0)$ consider the commutator

$$C(t) := F(t) \circ G(t) \circ (F(t))^{-1} \circ (G(t))^{-1}$$

Then $\dot{C}(0) = 0$. But $\tilde{C}(\tau) := C(\sqrt{\tau})$ is C^1 and

$$[f, g] := \frac{\partial}{\partial \tau}\Big|_{\tau=0} \tilde{C}(\tau) \in \mathfrak{g}.$$

It turns out that $[f,g]$ is bilinear, antisymmetric, with Jacobi identity: $0 = [f,[g,h]] + [h,[f,g]] + [g,[h,f]]$

Conversely: Any finite-dimensional linear space with such a bracket $[\cdot,\cdot]$ is called (abstract) Lie algebra $\tilde{\mathfrak{g}}$. Via a map exp one can construct a Lie group $\mathcal{G}$, such that $\tilde{\mathfrak{g}} = \mathfrak{g}$ = Lie algebra of $\mathcal{G}$