composition $\tilde{F}_m \circ \tilde{G} := T_m(\tilde{F} \circ \tilde{G})$

Definition: The Lie algebra $\mathfrak{g}$ of a Lie group $\mathcal{G}$ is the tangent space at $e = \mathrm{id} \in \mathcal{G}$, i.e. $\mathfrak{g} = T_e \mathcal{G}$.
In other words $f \in \mathfrak{g}$ are the tangents $f = \frac{\partial}{\partial t}\big|_{t=0} F(t)$ of smooth paths $t \to F(t) \in \mathcal{G}$ at $F(0) = e$

Note: Fix $H_0 \in \mathcal{G}$ and consider $F \longmapsto F \circ H_0$ via $\frac{\partial}{\partial t}\big|_{t=0}$ this defines a map $\mathfrak{g} = T_e\mathcal{G} \ni f \longmapsto f \circ H_0 \in T_{H_0}\mathcal{G}$

$\circ H_0 : f \longmapsto f \circ H_0$, $(H_0 \circ)' : f \longmapsto H_0' \cdot f$

Brief excursion: commutators and Lie-bracket.

For curves $F(t), G(t)$ through e at $t = 0$ and $f := \dot{F}(0)$, $g := \dot{G}(0)$ consider the commutator

$C(t) := F(t) \circ G(t) \circ (F(t))^{-1} \circ (G(t))^{-1}$

Then $\dot{C}(0) = 0$. But $\tilde{C}(\tau) := C(\sqrt{\tau})$ is C^1 and

$[f, g] := \frac{\partial}{\partial \tau}\big|_{\tau=0} \tilde{C}(\tau) \in \mathfrak{g}$.

It turns out that $[f,g]$ is bilinear, antisymmetric, with Jacobi identity: $0 \overset{!}{=} [f,[g,h]] + [h,[f,g]] + [g,[h,f]]$

Conversely: Any finite-dimensional linear space with such a bracket $[\cdot,\cdot]$ is called (abstract) Lie algebra $\tilde{\mathcal{G}}$. Via a map exp one can construct a Lie group $\mathcal{G}$, such that $\tilde{\mathfrak{g}} = \mathfrak{g} =$ Lie algebra of $\mathcal{G}$

03.05.2011

4 Theorem (revisited) [Takens folklore] (p. 15)

Consider a C^∞ v-field $f = Ax + \ldots$, $m \geq 2$,

$\mathcal{F}(f,t) :=$ flow of $\dot{x} = f(x)$, $F = Bx + \ldots := \mathcal{F}(f,1)$, time-1 map

$B = e^A$

Let $g = Ax + \ldots$ be the ad A^T normal form of F and $G = Bx + \ldots := \mathcal{F}(g,1)$. Then G is a B^T normal form of F.

<u>In short:</u> time-1 map of normal form = normal form of time-1 map

$$\begin{array}{ccc} f & \xrightarrow{n.f.} & g \\ \mathcal{F}(\cdot,1) \downarrow & /// & \downarrow \mathcal{F}(\cdot,1) \\ F & \xrightarrow{n.f.} & G \end{array}$$

<u>More precisely:</u> we claim ~~generell A~~

(i) there ex. ad A^T normal form transformation $\Phi = x + \ldots \in C^\infty$ s. th. the n-th Taylor terms g_n satisfy $g_n \in \ker(ad_m A)$ $\quad 2 \le n \le m$ for the transformed v-field $g = \Phi^* f$

(ii) $G := \mathcal{F}(g,1)$ satisfies
$$T_m G = T_m(T_m \Phi^{-1} \circ T_m F \circ T_m \Phi)$$

$$\boxed{\begin{aligned} \tilde f := T_m f(x) &= m\text{-th Taylor polyn. of } f \text{ at } x \\ &= Ax + f_2 + \ldots + f_m \end{aligned}}$$

$$\boxed{\begin{aligned} ad_n A g_n &= A^T g(x) - g_n'(x) A^T x \\ g(x) &:= \Phi'(x)^{-1} f(\Phi(x)) \end{aligned}}$$

(iii) Further assume $A^T A = A A^T$

G is in B^T normal form, up to order m, i.e. $B^T T_m G = T_m G \circ B^T$

~~$(\tilde{i})$ Given $F = Bx + \ldots$, $B = e^A$, there ex. $f = Ax + \ldots$? s.th. $T_m F = T_m \mathcal{F}(1,1)$.~~

<u>Remark 1:</u> What if $\mathcal{F}(f,1)$ does not exist?

All statements are about Taylor expansions at 0; therefore we may cut off f to ($f \equiv 0$ outside some fixed nbhd of 0

<u>Remark 2:</u> The map $f \to F$ cannot be inverted (not even locally, to finite order). Trivial linear example:

For $\det B < 0$, there cannot be real A with $B = e^A$ (indeed: $\det B = \det e^A = e^{\operatorname{trace} A} > 0$.

Also ex. $B \in SL(2, \mathbb{R})$ which are not (real) exponents.

But even if we assume $B = e^A$ for given $F(x) = Bx + \ldots$ no compatible f may exist.

($\to$ subharmonic bifurcation)

Remark 3: Although a short and direct proof is possible, we indulge in a big detour to Lie groups, to put the theorem into perspective.

Definition: Let $F(t)$ denote the solution of the following ODE on $\mathcal{G}$:

$$(*)\quad \dot{F}(t) = f \circ F(t)$$

. Here $f \in \mathfrak{g}$ is fixed.

Define $\exp: \mathfrak{g} \times \mathbb{R} \to \mathcal{G}$

$$(f, t) \mapsto \exp(f \cdot t) := F(t)$$

This is called the exponential map

Proposition:

(i) Let $H_0 \in \mathcal{G}$, $\tilde{F}(t) := F(t) \circ H_0$, F with $(*)$.
Then $\tilde{F}(t)$ solves $(*)$ with $\tilde{F}(0) = H_0$. In particular the solution of $(*)$ is global for whatever f.

(ii) $\exp(f \cdot (t+s)) = \exp(f \cdot t) \circ \exp(f \cdot s)$

HOMEWORK: $\exp(f \cdot (ts)) = \exp((s \cdot f) \cdot t)$

In particular we may define $\exp(f) := \exp(f \cdot 1)$

$\exp: \mathfrak{g} \to \mathcal{G}$. Then $\exp(f \cdot s) = \exp(s \cdot f)$ via $t = 1$.

(iii) Let $\Phi \in \mathcal{G}$, $f \in \mathfrak{g}$, $t \in \mathbb{R}$. Then $\Phi^{-1} \circ \exp(f \cdot t) \circ \Phi$ $(**)$
$= \exp((\Phi^* f) \cdot t)$ where $g = \Phi^* f := (\Phi')^{-1} \cdot (f \circ \Phi)$
$= ((\Phi^{-1})' f) \circ \Phi$

Proof (of the proposition):

(i) $\frac{d}{dt} \tilde{F}(t) = \frac{d}{dt}(F(t) \circ H_0) = \dot{F}(t) \circ H_0 \underset{(*)}{=} (f \circ F(t)) \circ H_0$

$= f \circ (F(t) \circ H_0) = f \circ \tilde{F}(t)$ solves $(*)$

(ii) Let $H_0 := F(s) = \exp(f \cdot s)$, $\tilde{F}(t)$, $\tilde{F}(t) := F(t) \circ H_0$
Then $\tilde{F}(t)$ solves $(*)$ with $\tilde{F}(0) = H_0 = F(s)$

(see (ii))

But so does $F(t+s)$. Therefore $\exp(f\cdot(t+s))$

$$= F(t+s) = \tilde{F}(t) = F(t)\cdot \underbrace{F(s)}_{H_0} = \exp(f\cdot t)\circ \exp(f\cdot s)$$

(iii) Let $\tilde{F}(t) := \Phi^{-1}\circ \underbrace{\exp(f\cdot t)}_{F(t)} \circ \Phi$ obviously a flow

$$\boxed{\begin{array}{l} \Phi^{-1}\circ\Phi = e \\ (\Phi^{-1})'\circ\Phi = (\Phi')^{-1} \end{array}}$$

Recall:

$(\Phi^{-1})'\circ\Phi = (\Phi^{-1})^{-1}$. Therefore

$$\frac{d}{dt}\tilde{F}(t) = \big(\underbrace{(\Phi^{-1})'\circ\underbrace{\Phi\circ\Phi^{-1}}_{\text{smuggle}}\circ F(t)\circ\Phi}\big)\cdot\big(\dot{F}(t)\circ\Phi\big)$$

(p.19)

$$= \underline{\big((\Phi')^{-1}\cdot \tilde{F}(t)\big)}\cdot\big(f\circ\underbrace{\Phi\circ\Phi^{-1}}_{\text{smuggle}}\circ F\circ\Phi\big)$$

$$= \underbrace{\big((\Phi')^{-1}\cdot(f\circ\Phi)\big)}\circ\tilde{F}(t)$$

$$=: g = \Phi^* f \qquad \square$$

Remark 1: For matrices, multiplication, this exp is the usual exponentiation: with $A := f$, $\dot{F} = AF$, $F(0) = id$, we have $F(t) = e^{At} = \exp(f\cdot t)$

matrices

Proof of Theorem 6.6.1

The key-observation is that $T_m\,\mathcal{F}(f,t) = \exp(T_m f\cdot t)$ (***) in the Lie group $\mathcal{G} := \mathrm{Diff}_m = \{F(x) = Bx + \ldots + F_m,\ B\in GL(N,\mathbb{R})\}$ with Lie algebra $\mathfrak{g} := \mathrm{diff}_m = \{f(x) = Ax + \ldots + f_m,\ \underbrace{A\in M(N,\mathbb{R})}_{\text{any matrix}}\}$

To prove (***) we first abbreviate

$T_m F$, $T_m f$, $T_m G$, $T_m g$, $T_m\Phi$, ... by $\tilde{F}, \tilde{f}, \tilde{G}, \tilde{g}, \tilde{\Phi}$.

Claim: $\tilde{\mathcal{F}}(f,t) = \exp_m(\tilde{f}\cdot t)$ solves

$$\frac{\partial}{\partial t}\tilde{\mathcal{F}}(\tilde{f},t)(x_0) = \tilde{f}_0\,\tilde{\mathcal{F}}(\tilde{f},t)(x_0) \quad \text{for: } \tilde{\mathcal{F}}(\tilde{f},0) = e = id$$

Consider $x(t) := \mathcal{F}(\tilde{f},t)(x_0)$

Then $\dot{x}(t) = \tilde{f}(x(t))$, $x(0) = x_0$

$$\frac{\partial}{\partial t} T_m \mathcal{F}(\tilde f, t)\cancel{(x_0)} = T_m(\tilde f(T_m \mathcal{F}(\tilde f, t)(x_0)))$$
$$= \tilde f \underset{\substack{\uparrow \\ \text{composition from } \mathcal{G}}}{\circ_m} T_m \mathcal{F}(\tilde f, t)\cancel{(x_0)}$$

as was claimed.

To prove claim (i), we recall $\boxed{6.3}$

Claim (ii) says that $\tilde G = \tilde\Phi^{-1} \circ_m \tilde F \circ_m \tilde \Phi$

But this follows from $\boxed{6.2}$ Proposition, (iii) (p.):

$$\tilde\Phi^{-1} \circ_m \underbrace{\tilde{\mathcal{F}}(\tilde f, t)}_{\tilde F} \circ_m \tilde\Phi \underset{\substack{(**)\\(p\,19)}}{=} \tilde{\mathcal{F}}(\underbrace{\tilde\Phi^* \tilde f}_{=\tilde g}, t) = \underbrace{\tilde{\mathcal{F}}(\tilde g, t)}_{\tilde G}$$

$\underset{t=1}{\Rightarrow}$ claim(ii)

Here we used that $\Phi^* f = g =: \Phi'(x)^{-1} f(\Phi(x))$ indeed is, after truncation to level m, the expression $\tilde\Phi^* \tilde f = (\tilde\Phi')^{-1} \cdot (\tilde f \circ_m \tilde\Phi)$ which we derived abstractly in the Lie group setting!

(iii) For $A^T A = A A^T$ we already observed in $\boxed{6.3}$ that $g \in \ker(\operatorname{ad} A^T)$ (including linear term of $g = Ax + \dots$) implies $e^{-A^T t}\tilde{\mathcal{F}}(g,t) \circ \underbrace{e^{A^T t}}_{\in \Gamma = \operatorname{clos}\{e^{A^T t} \mid t \in \mathbb{R}\}} = \tilde{\mathcal{F}}(g, t) =$

This implies, for $t=1$, that $B^{-T} \circ \tilde G \circ B^T = \tilde G$ i.e. $\tilde G$ is in B^T normal form. $\square$

<u>Remark:</u> Actually we had a lemma which said $\tilde g \in \ker(\operatorname{ad} A^T)$ implies $\underbrace{e^{-A^T t} \circ \tilde g \circ e^{A^T t}}_{\tilde g_t} = \tilde g$

$$\tilde g_t = (\underbrace{e^{A^T t}}_{\Phi})^* \underbrace{\tilde g}_{f}$$

define: Φ f

But this implies $\Phi^{-1} \circ_m \exp(\tilde g \cdot t)\Phi$
$$= e^{-A^T t} \circ \exp_m(\tilde g t) \cdot e^{A^T t}$$
$$= \exp_m(\underbrace{(e^{A^T t})^* \tilde g}_{\tilde g} \cdot t) = \exp_m(\tilde g \cdot t) \underset{t=1}{=} \tilde G$$

$$e^{-A^T \cancel{t}} \cdot \tilde G \circ e^{A^T \cancel{t}}$$

<u>Remark 2:</u> We have prooved that it is one and the same transformation Φ which

(a) puts $\tilde{f}$ into normal form $\tilde{g} := \tilde{\Phi}^* \tilde{f}$ and
(b) puts $\tilde{F}$ into normal form $\tilde{G} \circ \tilde{F} \circ \tilde{\Phi}$

Remark 3: Fix $\Phi \in \mathcal{G}$, general Lie-group.

Then
$$\tilde{\rho}(\Phi) : \mathfrak{g} \to \mathfrak{g}, \quad f \to \Phi^* f \qquad \text{is linear.}$$

Moreover $(\Phi \circ \Psi)^* = \Psi^* \circ \Phi^*$

Therefore $\rho(\Phi) = (\tilde{\rho}(\Phi))^{-1}$ is a group homomorphism

$$\rho : \mathcal{G} \to (\text{Matrices on } \underbrace{\mathfrak{g}}_{= GL(\mathfrak{g})}) \qquad \Phi \longmapsto \rho(\Phi)$$

check: $\rho(\Phi \circ \Psi) := ((\Phi \circ \Psi)^*)^{-1} = (\Psi^* \circ \Phi^*)^{-1}$
$= (\Phi^*)^{-1} \circ (\Psi^*)^{-1} = \rho(\Phi) \circ \rho(\Psi)$

In Lie theory $\rho(\Phi) = Ad(\Phi)$ is called the adjoint representation of $\mathcal{G}$ on $\mathfrak{g}$

Now consider the special case $\Phi = \exp(g \cdot t)$, for some $g \in \mathfrak{g}$. Then the "infinitisimal version" of Ad,

$$ad(g) f := \frac{d}{dt}\Big|_{t=0} \Phi(-t)^* f = [g, f]$$

is the Lie bracket. Our Lemma on h_t said:

$$ad(A) f = [A, f] = 0 \quad \Rightarrow \quad \underbrace{Ad(e^{At}) f}_{= (e^{-At})^* f} \equiv f$$

It turns out that

$ad : \mathfrak{g} \to M(N) = gl(\mathfrak{g})$ is a homomorphism of Lie algebras, where we simply have

$[A, B] := AB - BA$ in $GL(\mathfrak{g})$ (Jacobi identity)