5

7. Bifurcation

Die Bifurkationstheorie untersucht, technisch gesprochen, was passiert, wenn der Satz über implizite Funktionen scheitert. Thematisch geht es vor allem um das Studium dyn. Systeme

$$\dot{x} = f(\lambda, x), \qquad x_{n+1} = F(\lambda, x_n) \qquad (\lambda = \text{const})$$

die von einem oder mehreren Parametern $\lambda \in \Lambda$ = Banach, $\mathbb{R}^m$, $\mathbb{R}$ abhängen.

Parameter: alles woran man drehen kann (oder möchte, oder leider nicht kann ...)

7.1 Lyapunov-Schmidt Reduction

Goal: Solve $\mathcal{F}(w) = 0$ nahe $\mathcal{F}(0) = 0$ for

$$\mathcal{F}: W^{\text{Banach}} \longrightarrow Z^{\text{Banach}}, \quad C^k, \quad k \geq 1$$

Assumption: Let $\mathcal{L} = \mathcal{F}'(0): W \to Z$, bounded, lin., be quasi-Fredholm

Definition: $\mathcal{L}$ beschr., lin., is called quasi-Fredholm if $\mathcal{L}^{-1}(0) = \operatorname{Ker} \mathcal{L}$ and $\operatorname{Range}\mathcal{L}$ are both closed with closed complement:

$$W = \operatorname{Ker} \mathcal{L} \oplus (\operatorname{Ker} \mathcal{L})^{c\,(\text{complement})}, \quad Z = \operatorname{Range} \mathcal{L} \oplus (\operatorname{Range} \mathcal{L})^c$$

We call $\mathcal{L}$ Fredholm if $\dim \ker \mathcal{L}$, $\operatorname{codim} \operatorname{Range} \mathcal{L} = \dim (\operatorname{Range} \mathcal{L})^c$ are both finite

Fredholm-index

$$\operatorname{ind}(\mathcal{L}) := \dim \ker \mathcal{L} - \operatorname{codim} \operatorname{Range} \mathcal{L}$$
$$= \text{"deficiency"} - \text{"corank"}$$

Proposition (see [Kato])

(i) The set of Fredholm operators $\mathcal{L}$ of index $i \in \mathbb{Z}$ is open in the set of bd. lin. operators $W \to Z$, for any fixed i;

(ii) $\mathcal{L}$ Fredholm, $\mathcal{K}: W \to Z$ compact (i.e. $\mathcal{K}$ (unit ball) = rel. cpt.) $\Rightarrow \mathcal{L}+\mathcal{K}$ Fredholm
$\operatorname{ind}(\mathcal{L}+\mathcal{K}) = \operatorname{ind}(\mathcal{L})$

(iii) $\mathcal{L}_j : W_j \to W_{j+1}$ Fredh
$\Rightarrow \mathcal{L}_2 \circ \mathcal{L}_1 : W_1 \to W_3$ Fredholm,
$\operatorname{ind}(\mathcal{L}_2 \cdot \mathcal{L}_1) = \operatorname{ind}(\mathcal{L}_2) + \operatorname{ind}(\mathcal{L}_1)$

$\boxed{\begin{array}{l} id : W \to W \\ Z := W \oplus \mathbb{R}^m \\ i : W \to Z = W \oplus \mathbb{R}^m \\ \quad w \mapsto w \\ \operatorname{ind}(i) = -m \end{array}}$

Remarks:

① $id: W \to W$ has $\operatorname{ind}(id) = 0$
$id + \mathcal{K}^{cpt} : W \to W$ has $\operatorname{ind}(id + \mathcal{K}) = 0$

Fredholm: $\underbrace{x(t)}_{(id \cdot x)(t)} = \underbrace{\int_0^t k(t,s)\, x(s)\, ds}_{(\mathcal{K} x)(t)} + g(t)$

$\Leftrightarrow \quad (id - \mathcal{K})\, x = g$

"Fredholm alternative"

solution x ex. for all g $\Leftrightarrow$ $g = 0$ has only trivial solution $x = 0$

On suitable Banach space $id - \mathcal{K} : W \to W$ this is clear:
$\operatorname{ind}(id - \mathcal{K}) = \operatorname{ind}(id) = 0$, hence $id - \mathcal{K}$ surjective $\Leftrightarrow$ $id - \mathcal{K}$ injective

② For $W = \mathbb{R}^m$, $Z = \mathbb{R}^n$ finite-dimensional, all linear operators are bd., cpt. and Fredholm:
$\operatorname{ind}(L) = \operatorname{ind}(0) = m - n$

③ $\ker \mathcal{L} = \mathcal{L}^{-1}(0)$ is always closed, by continuity of $\mathcal{L}$. Closedness of Range $\mathcal{L}$ is a serious assumption.

Finite-dimensional: subspaces $V = \operatorname{span}\{v_1, \dots, v_n\}$ of Banach spaces X always possess a closed complement

([Hahn-Banach] Thm. of F. A.:

for $v \in X$ ex. $f \in X^*$ s.th. $f(v) \neq 0$.

Take $\ker(f)$ as complement to $\langle v \rangle$

$\ker f_1 \cap \dots \cap \ker f_n$ as complement to $\langle v_1, \dots, v_n \rangle$ $\square$

To solve $\mathcal{F}(w) = 0$ for quasi-Fredholm

$\mathcal{L} = \mathcal{F}'(0)$ we decompose

$$w = u \oplus v \underset{\text{by abuse}}{=} (u,v) \quad \text{, via} \quad W = \underbrace{\ker \mathcal{L}}_{u} \oplus \underbrace{\ker \mathcal{L}^c}_{v}$$

Similarly let P, $Q := id - P$ denote cont. projections onto $\operatorname{Range} \mathcal{L}$, $(\operatorname{Range} \mathcal{L})^c$, respectively, according to

$$Z = \underbrace{\operatorname{Range} \mathcal{L}}_{\operatorname{Range} P} \oplus \underbrace{(\operatorname{Range} \mathcal{L})^c}_{\operatorname{Range} Q}$$

Note: $\mathcal{F}(w) = 0 \Leftrightarrow \begin{cases} 0 = P\mathcal{F}(w) = P\mathcal{F}(u,v) \overset{!!}{=} G(u,v) \\ 0 = Q\mathcal{F}(w) \\ \quad = Q\mathcal{F}(u,v) \end{cases}$

$$\begin{array}{c|cc} & u & v \\ \hline \end{array} \qquad \mathcal{L} = \begin{array}{c} PZ \\ QZ \end{array} \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

$P^2 = P$
$\Rightarrow Q^2 = Q$
$QP = PQ = 0$

Study $G(u,v) = 0$

We know $G(0,0) = 0$

We also know $\underset{\substack{\uparrow \\ \text{partial derivative}}}{D_v} G(0,0) : (\ker \mathcal{L})^c \to PZ$

$v \mapsto P\mathcal{L}(v) = \mathcal{L}(v)$

& bd., lin., injective, surjective

By "closed graph theorem" aka "open mapping principle" aka "theorem of the bounded inverse" from F.A.

(→ [Kato]) bd. + lin. + bij. ⇒ the inverse also <u>bd.</u> lin

By usual implicit function theorem applied to G, we get unique solution $v = V(u)$ for $G(u,v) = 0$ locally, in $(u,v) \approx (0,0)$. Moreover $V \in C^k$

$$V(0) = 0 \quad \text{and} \quad V'(0) = -(D_v G)^{-1} \underbrace{D_u G}_{P\mathcal{L}|_{\ker \mathcal{L}}} \Big|_{u=v=0} = 0$$

$$\left.P\mathcal{L}|_{\ker\mathcal{L}}\right\} = 0$$

$$\mathcal{F}(w) = 0 \underset{\text{loc.}}{\Longleftrightarrow} v = V(u), \; V(0) = 0, \; V'(0) = 0$$

$V: \ker \mathcal{L} \to (\ker \mathcal{L})^c$, C^k

<u>and</u> $\Phi(u) := Q\,\mathcal{F}(u, V(u)) = 0$, $\quad \Phi: \ker\mathcal{L} \to (\text{Range } \mathcal{L})^c$

The rhs. is called Lypunov-Schmidt-reduction (LS) with the LS reduced equation $\Phi(u) = 0$.

<u>Theorem:</u> Let $\mathcal{F}: W \to Z$ be C^k, $k \geq 1$, $\mathcal{F}(0) = 0$, $\mathcal{L} := \mathcal{F}'(0)$ quasi-Fredholm. Then there ex. a nbhd. of $w = 0$ in W, where $\mathcal{F}(0) = 0$ is equivalent to the LS-reduced equation $\Phi(u) := Q\mathcal{F}(u, V(u)) = 0$ with $V(u)$ defined as above.

Moreover $\quad V(0) = 0, \quad V'(0) = 0$

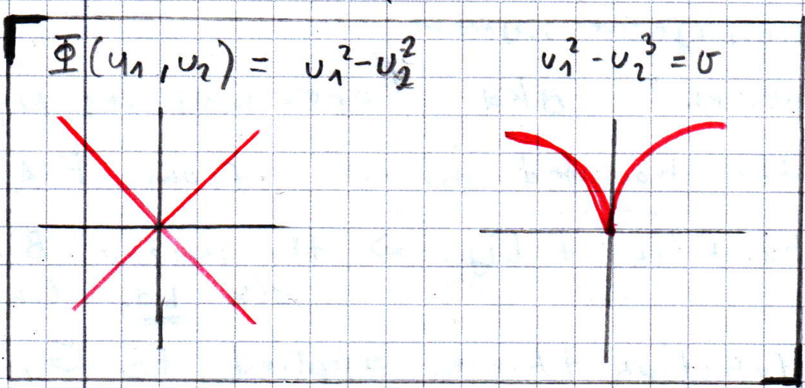
$$\Phi(0) = 0, \quad \Phi'(0) = 0.$$

<u>Proof:</u> It remains to check the claims on Φ

$$\Phi(0) = Q\,\mathcal{F}(u, V(u))\big|_{u=0} = Q\, \underbrace{\mathcal{F}(0, \underbrace{V(0)}_{=0})}_{=0} = 0$$

$$\Phi'(0) = Q\, \underbrace{D\mathcal{F}\cdot(\ldots)}_{\mathcal{L}}\big|_{u=v=0} = 0 \quad \text{by definition of } Q$$

$\square$



7.2 Stationary bifurcation

Consider $0 = f(\lambda, x)$, $x \in X^{\text{Banach}}$, $f: \mathbb{R} \times X \to Y^{\text{Banach}}$, $(\lambda, x) \mapsto f(\lambda, x)$

and assume $f \in C^k$, $k \geq 2$

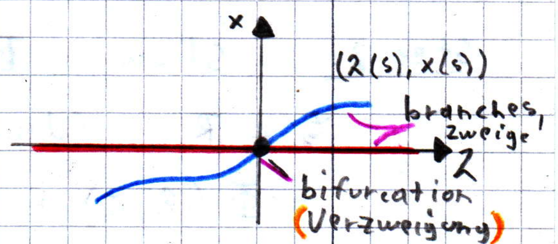
Let $f(0,0) = 0$ and assume $L := D_x f(0,0) : X \to Y$

is Fredholm of index 0. [Crandall, Rabinowitz] (1971)

Case 1: $\operatorname{Ker} D_x f(0,0) = \{0\}$, trivial kernel.

The the usual IFT provides solutions $x = x(\lambda)$, locally. Indeed $\operatorname{ind}(L) = 0$, $\operatorname{Ker} L = \{0\} \Rightarrow$ $\operatorname{Range}(L) = Y$ and bd. inverse L^{-1} exists.

Case 2: $\ker L \neq \{0\}$ „nontrivial"



Theorem:

For f as above assume a trivial solution $f(\lambda, 0) = 0$ for all λ. Further suppose

(i) $\dim \ker D_x f(0,0) = 1$ and

(ii) $D_\lambda D_x f(0,0)|_{\operatorname{Ker} L} \not\subset \operatorname{Range} L$ (i.e. not a subspace)

Then the zero set of f near $(0,0)$ consists of the trivial branch $(\lambda, 0)$ and a nontrivial C^{k-1}-branch $\mathbb{R} \ni s \longmapsto (\lambda(s), x(s))$ through $\lambda(0) = 0$, $x(0) = 0$, $\dot{x}(0) \neq 0$. The two branches are each unique up to re-parametrization.

Proof-plan:

① LS-reduction of $f = 0$ to $\Phi(\lambda, \xi) = 0$

$\xi \in \operatorname{Ker} L$, $\Phi \in C^k$

② $\Phi(\lambda, \xi) = \xi \cdot \varphi(\lambda, \xi)$, $\varphi \in C^{k-1}$

③ $\varphi(0,0) = 0$ (Here we use $\dim \xi = \dim \Phi = \dim \varphi = 1$) $\lambda = \lambda(\xi)$.

④ $D_\lambda \varphi(0,0) \neq 0$

Then we are done by IFT for $\varphi(\lambda, \xi = 0)$

Proof:

$f(\lambda,0)=0$
$\Rightarrow D_\lambda f(0,0)=0$

Step 1: $W := \mathbb{R}\times X$, $Z := Y$,

$\mathcal{F}(w) := f(\lambda,x)$

$\mathcal{L} := \mathcal{F}'(0) = (D_\lambda f, D_x f)|_{\lambda=0, x=0}$

$= (0,L)$, Fredholm,

index $\mathcal{L} = 1 +$ index $L = 1$

Therefore $\mathcal{F}(\lambda,x)=0 \Leftrightarrow x = \xi \oplus \underbrace{V(\lambda,\xi)}_{\ker \mathcal{L}}$ and

$\underbrace{\Phi(\lambda,\xi)}_{4} = Q f(\lambda, \xi + V(\lambda,\xi))$

$QL=0$
$\xi \in \ker L$

Step 2: show $\Phi(\lambda,0)=0$ $(\forall\lambda)$ then indeed

$\xi\in\mathbb{R}$ $(\dim \ker L = 1)$ we have $\Phi(\lambda,\xi)$

$= \underbrace{\Phi(\lambda,0)} + \int_0^1 \frac{d}{d\vartheta}\Phi(\lambda,\vartheta\xi)\,d\vartheta$

$= 0 + \underbrace{\left(\int_0^1 D_2\Phi(\lambda,\vartheta\xi)\,d\vartheta\right)\xi}_{\varphi(\lambda,\xi)}$

C^{k-1} for $\Phi\in C^k$

To show $\Phi(\lambda,0)=0$

observe that $\mathcal{F}(\lambda, \underset{(\xi,v)}{\overset{\|}{0}}) = 0 \Rightarrow G(\lambda, \overset{\xi\ v}{0,0}) = 0$

$V(\lambda,0)=0$

$\Rightarrow \Phi(\lambda,0) = Q\mathcal{F}(\lambda,0,V(\lambda,0)) = Qf(\lambda,0)=0$

Step 3: $\varphi(\lambda,0) = D_\xi|_{\xi=0}\Phi(\lambda,0) = 0$ at $\lambda=0$ by LS

Step 4: $D_\lambda\varphi(0,0) = D_\lambda D_\xi|_{\lambda=0,\xi=0}\Phi(\lambda,\xi)$

$= D_\lambda D_\xi Q\mathcal{F}(0) = D_\lambda D_\xi Q f(0,0)$

$= Q\underbrace{D_\lambda D_x f(0,0)|_{\ker L}}_{\notin \text{Range } L} \neq 0$

Q "kills" Range L, only.

Remark 1: Meaning of condition (ii)

Consider $X = Y = \mathbb{R}^N$. Then $f(\lambda, x) = L(\lambda)x + \dots$

$L(\lambda) := D_x f(\lambda, 0)$

and L has 1-dim kernel span $\langle \xi \rangle$.

Assume, in addition, that $\mu_0 = 0$ is a alg. simple eigenvalue of $L(0) =: L$.

(Note: e-vector $= \xi_0$)

Then μ_0 continues to eigenvalue $\mu(\lambda)$ and ξ_0 to eigenvector $\xi(\lambda)$ for small $\lambda \neq 0$.

$\mu(\lambda)\xi(\lambda) = L(\lambda)\xi(\lambda)$ all C^{k-1} in λ

$$\frac{d}{d\lambda}\Big|_{\lambda=0}: \quad \mu' \cdot \xi_0 \cdot \mu_0 \xi_0' = L(0)\xi_0' + L'(0)\xi_0$$

Let Q be the eigenprojection of L onto $\langle \xi_0 \rangle$

then $QL = 0$ $\quad (\mathbb{R}^N = \langle \xi_0 \rangle \oplus \text{Range } L(0))$

$$\mu'(0) \cdot \underbrace{Q\xi_0}_{\xi_0} = \underbrace{QL}_{=0}\xi_0' + QL\xi_0$$

$$= Q D_\lambda D_x f(0,0)\big|_{\ker L(0)}$$

is equivalent to $\mu'|_{\lambda=0} \neq 0$

Condition (ii) (p.) therefore means that the eigenvalue $\lambda \longmapsto \mu(\lambda)$ crosses $\mu_0 = \mu(0) = 0$ with non vanishing speed $\mu'(0)$, as λ increases through $\lambda = 0$.

$\Rightarrow$ Transverse crossing condition.

$\lambda = 0,\ x = 0$

Remark 2: Compare situation of Remark 1 with center mf./normal form reduction, for $\dot{x} = f(\lambda, x)$.

Trick: add $\dot{\lambda} = 0$. Assume $\mu_0 = 0$ is the only pure imaginary eigenvalue of L. Then cmf. has dim $= 2$, tangent space $\mathbb{R} \times \text{span}(\xi_0) \ni (\lambda, \xi)$

and reduced flow

$$\dot{\xi}_3 = \tilde{\Phi}(\lambda, \xi_3) \quad , \qquad \dot{\lambda} = 0$$

Again $\tilde{\Phi}(\lambda, 0) = 0$ (equilibria $\in$ cmf.), hence

$$\tilde{\Phi}(\lambda, \xi_3) = \xi_3 \cdot \tilde{\varphi}(\lambda, \xi_3),$$

$$\dot{\xi}_3 = \xi_3 \, \tilde{\varphi}(\lambda, \xi_3)$$

Normal form? All is kernel here, $L = 0$, $\mathrm{ad}_M L^T = 0$

$\Rightarrow$ RATS

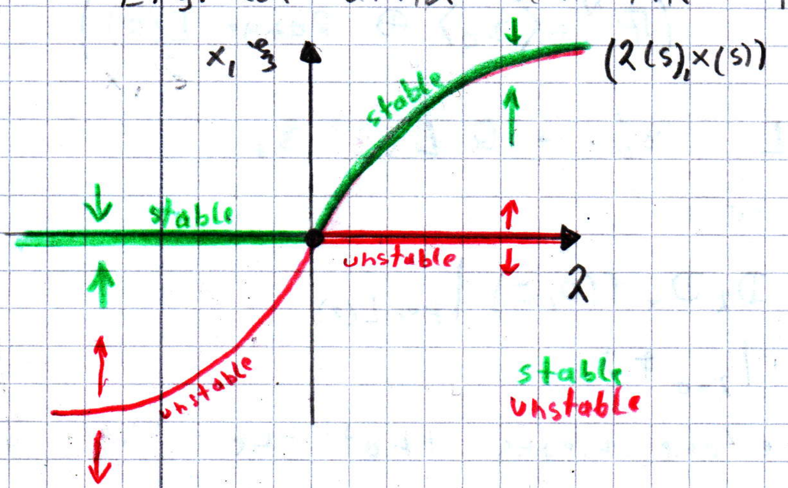
But we know all equilibria, through LS:

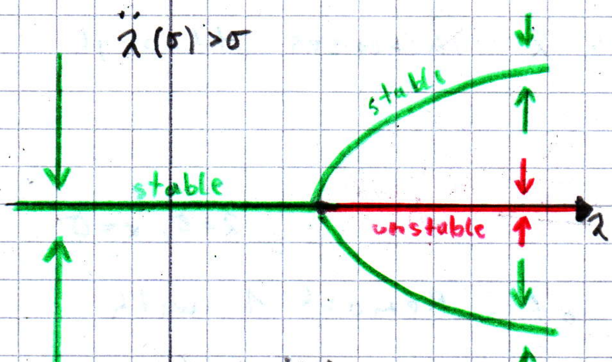
$$\tilde{\varphi}(\lambda, \xi_3) = 0, \; \xi_3 \neq 0 \iff \varphi(\lambda, \xi_3) = 0, \; \xi_3 \neq 0$$

Because $\dim \xi_3 = 1$ we can now draw dynamic conclusions concerning stability of equilibria (if $\mathrm{spec}\, L \subseteq \{\mathrm{Re} < 0\} \cup \{0\}$ from cmf.

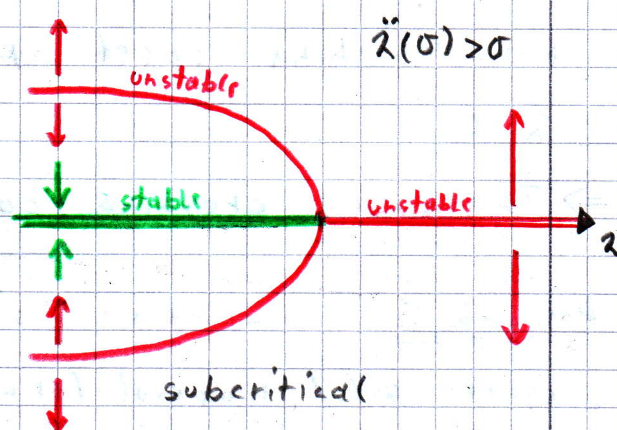
E.g. we arrive at the following bifurcation diagrams

e.g. for $\varphi'(0) > 0$





supercritical



subcritical

pitchfork bifurcation (for $\dot{\lambda}(0) = 0$, $\ddot{\lambda}(0) \neq 0$