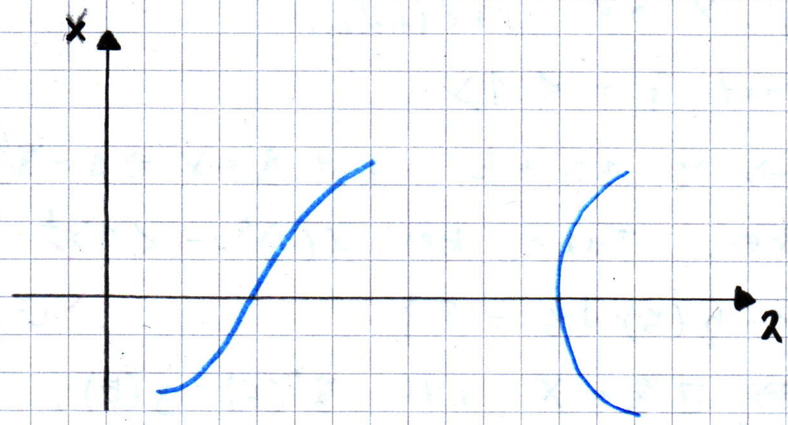
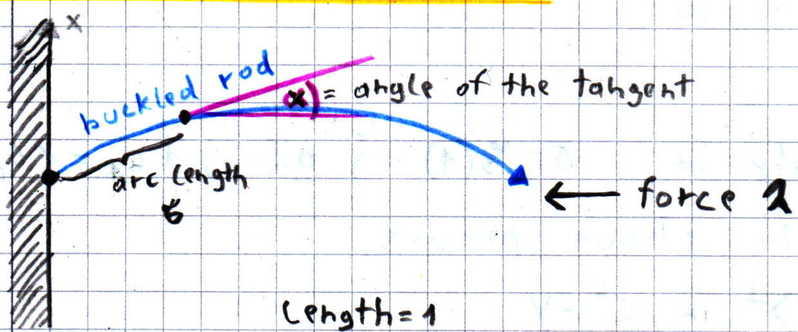
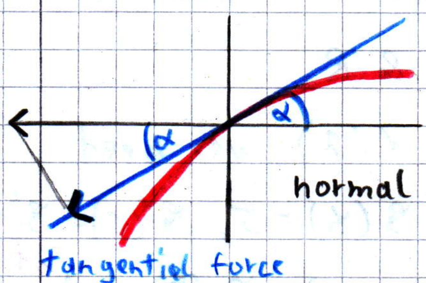


$\dot{x} = f(\lambda, x)$, $x \in \mathbb{R}^N$, center manifold $\Rightarrow$ exchange of stability

Example 1 : Eulers rod



elasticity model: $\boxed{\begin{aligned} & x''(\sigma) \sim -\lambda \sin x(\sigma) \\ & x''(\sigma) + \lambda \sin x(\sigma) = 0 \\ & x'(0) = x'(1) = 0 \end{aligned}}$ ⊛



normal force $= -\lambda \cdot \sin \alpha$

How to rewrite ⊛ as $f(\lambda, x) = 0$

$x(\cdot) \in X := C^2([0,1], \mathbb{R}) \cap \{x'(0) = x'(1) = 0\}$

$Y := C^0([0,1], \mathbb{R})$

$\|x\|_X = \|x\|_{C^0} + \|x'\|_{C^0} + \|x''\|_{C^0}$

Then $f: \mathbb{R} \times X \to Y$, C^∞

for $f(\lambda, x) := x''(\cdot) + \lambda \sin x(\cdot)$

Note: $f(\lambda, 0) = 0$

$\mathcal{L}(\lambda)\xi = D_x f(\lambda, 0)\xi = \xi''(\cdot) + \lambda \xi(\cdot)$

$\mathcal{L}(\lambda): X \to Y$ is linear

$\mathcal{L}(\lambda) = \left(\frac{\partial}{\partial \sigma}\right)^2 + \underbrace{\lambda \, \mathrm{id}}_{\text{compact!}}$

Arzela - Ascoli from C^2 to C^0

Therefore $\mathcal{L}(\lambda)$ Fredholm $\Leftrightarrow$ $\mathcal{L}(0) = \left(\frac{\partial}{\partial \tau}\right)^2$ Fredholm

Show $\mathcal{L}(0) = \left(\frac{\partial}{\partial \tau}\right)^2 : X \to Y$ Fredholm

(a) $\ker \mathcal{L}(0) = \{\text{constants}\} = \langle 1 \rangle$

indeed $\xi'' = 0 \Rightarrow \xi = a\tau + b$ but $\xi \in X \Rightarrow a = \xi'(0) = 0$

and $\xi = b$ is in ker. Take $\ker \mathcal{L}(0)^c := \langle 1 \rangle^\perp$ (L^2 orthogonal complement)

$= \{ \eta \mid \int_0^1 1 \cdot \eta(\tau)\, d\tau = 0 \}$

(b) $g \in \text{Range } \mathcal{L}(0) \Leftrightarrow \exists \xi \in X$ s.th. $\xi''(\tau) = g(\tau)$, $g \in C^0$

$\Rightarrow \xi'(\tau) = \underbrace{\xi'(0)}_{=0} + \int_0^\tau g(\tau)\, d\tau$

$\Rightarrow \xi(\tau) = \underbrace{\xi(0)}_{\substack{\text{constant} \\ \ker \mathcal{L}(0)}} + \int_0^{\tau_1} \left(\int_0^{\tau_2} g(\tau_2)\, d\tau_2 \right) d\tau_1$

However $\xi \in X$ only if $0 = \xi'(1) = \int_0^1 g(\tau_1)\, d\tau_1 = \langle 1, g \rangle_{L^2}$

Together, rereading the above proves

$\text{Range } \mathcal{L}(0) = \langle 1 \rangle^\perp$ in $C^0 = Y$

Therefore $\mathcal{L}(0) : X \to Y$ is Fredholm of index $= 0$

Therefore $\mathcal{L}(\lambda) : X \to Y$ is Fredholm of index $= 0$

Now check assumption (i), (ii) of C & R

(i) To obtain nontrivial kernel of $\mathcal{L}(\lambda)$ we need nontrivial solution ξ of $\xi''(\tau) + \lambda \xi(\tau) = 0$, $\xi'(0) = \xi'(1) = 0$ for given $\lambda > 0$. ode & $\xi'(0) = 0 \Rightarrow \xi(\tau) = A \cos(\sqrt{\lambda}\tau)$ $=1$

Therefore $\dim \mathcal{L}(\lambda) \leq 1$.

But to have nontrivial kernel element $\xi \in X$ we also require: $0 = \xi'(1) = -\sqrt{\lambda} \sin(\sqrt{\lambda})$, i.e. $\lambda = \lambda_k = (k\pi)^2$, $k = 1, 2, 3, \ldots$

These $(\lambda_k, 0)$ are candidates for bifurcation points

Let us check (ii),

$\forall g \in Y$: Qg = projection onto $(\text{Range } \mathcal{L}(\lambda_k))^c$

$\ker \mathcal{L}(\lambda_k) = \langle \cos(\sqrt{\lambda_k}\tau) \rangle \underset{\uparrow \text{CHECK}}{=} (\text{Range } \mathcal{L}(\lambda_k))^\perp$

(self adjointness of $\mathcal{L}(\lambda)$)

Q = orthog. proj. onto $\cos(\sqrt{\lambda_k}\,\sigma)$, i.e.

$$Qg = 2\left(\int_0^1 (\cos(k\pi\sigma))\, g(\sigma)\, d\sigma\right)\cdot \cos(k\pi\sigma)$$

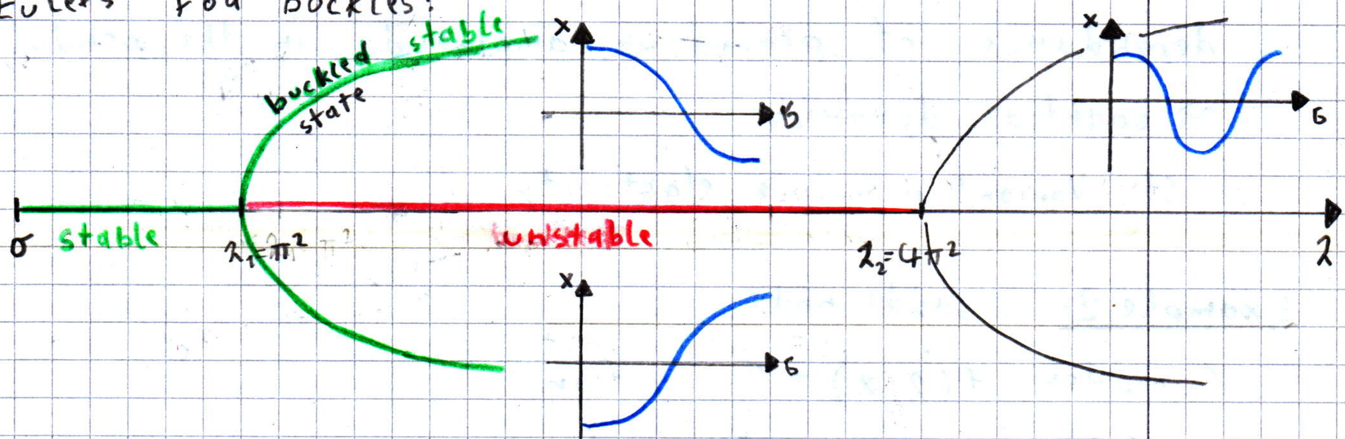
$$\Rightarrow Q\, D_\lambda D_x f(0,\sigma) \underbrace{\cos(k\pi\sigma)}_{\xi \in \operatorname{Ker} \mathcal{L}(\lambda)} = Q\, D_\lambda (\xi'' + \lambda \xi) = Q\xi$$

$$= 2\left(\int_0^1 \cos(k\pi\sigma)\cdot\cos(k\pi\sigma)\, d\sigma\right)\cos(k\pi\sigma)$$

$$= 1 \cos(k\pi\sigma) \neq 0$$

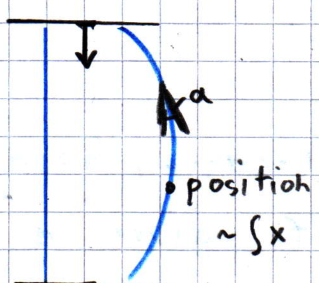
Now C & R show us how:

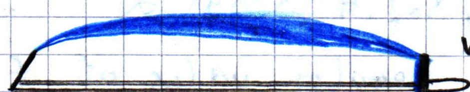
Euler's rod buckles:



Since $f(\lambda) = -x = -f(\lambda, x)$, the bifurcation branches are not transcritical but symmetric to the λ-axis. Further calculation exhibts supercritical pitchforks.

Buckled rod should look like:





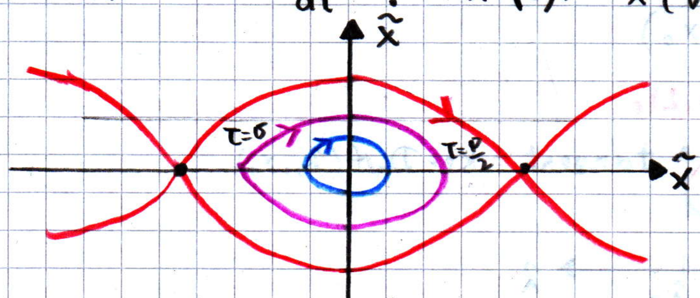
violin bow

Alternatively, we could view ⊛ as a pendulum

Let $\tilde{x}(\sqrt{\lambda}\,\sigma) := x(\sigma)$. Then

$$\ddot{\tilde{x}}\cdot\lambda + \lambda \sin\tilde{x} = x'' + \lambda \sin x = 0$$

with $\cdot = \frac{d}{dt}$, $\dot{\tilde{x}}(0) = \dot{\tilde{x}}(\sqrt{\lambda}) = 0$ $\qquad \ddot{\tilde{x}} + \sin\tilde{x} = 0$



Therefore solutions of $\tilde{x}$ with minimal period p

provide solutions x

$\sqrt{\lambda} = k \cdot \frac{p}{2}$ i.e.: $\lambda = k^2 \frac{1}{4} p^2$

p: as a function of amplitude is called the period map ("time map") [⇝ Carlos Rocha].

For amplitude $\searrow 0$ we have $p \to 2\pi$, hence $\lambda \to (k\pi)^2 \lambda_k$

The dependence $\lambda(\gamma), x(\gamma)$ of the bifurcating Euler branches are all related to one and the same dependance of period on amplitude, in the pendulum

⇝ spartial dynamics

St. Variant principle elasticity

Example 2: sadde node

Consider $f(\lambda, x) = 0$ for

$f: \mathbb{R} \times X \to Y$ C^k, $k \geq 1$

Do not assume any "trivial branch". Instead suppose

$f(\lambda_0, x_0) = 0$

$D_x f(\lambda_0, x_0) = \mathcal{L} =$ Fredholm, index $= 0$

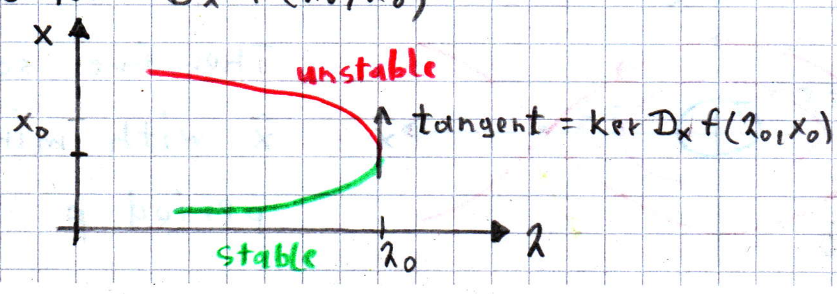
i) $\dim \operatorname{Ker} \mathcal{L} = 1$

ii) $D_\lambda f(\lambda_0, x_0) \notin \operatorname{Range} \mathcal{L}$

Then 0 is a regular value of f, locally, because $\operatorname{Range} D_{(\lambda,x)} f = Y$ LS for $\mathcal{F} := f$ gives a 0-dim LS reduced equation $\phi = 0$ i.e. nothing to solve there.

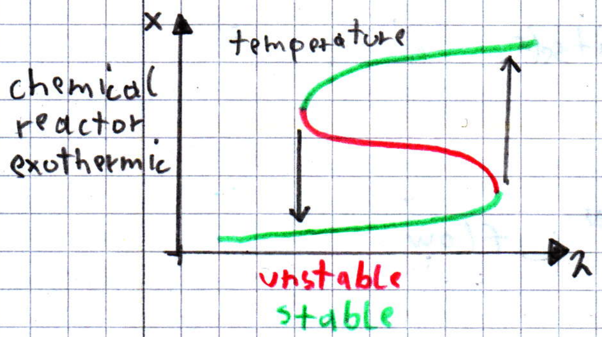
Solutions are therefore given by $v = V(u)$ alone, i.e. $\lambda = \lambda(s)$, $x = x(s)$ and $\lambda'(0) = 0$, $\lambda_0 = \lambda(0)$, $x_0 = x(0)$,

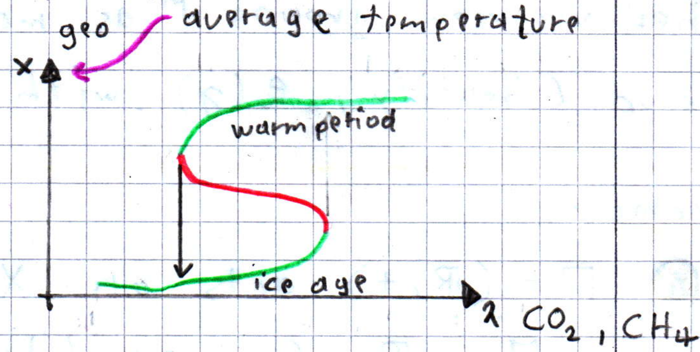
$x = x(s) \in \operatorname{Ker} D_x f(\lambda_0, x_0)$



For $\dot{x} = f(\lambda, x) \in X = \mathbb{R}^N$, $Y = \mathbb{R}^N$ we can invoke cmf again and obtain, e.g., stability analysis.

Global examples:





hysteresis. slow increase / decrease of λ on lower/upper branch.

7.3 Stationary symmetry breaking

Let Γ be a group which acts on a set X; i.e we have a map

$$\varrho : \Gamma \times X \to X \qquad \varrho(id) = id$$
$$\varrho(\gamma^{-1}) = \varrho(\gamma)^{-1}$$
$$(\gamma, x) \mapsto \varrho(\gamma, x) = \varrho(\gamma)x = \gamma x$$

such that $\varrho(\gamma_1) \circ \varrho(\gamma_2) = \varrho(\gamma_1 \cdot \gamma_2)$

For any $x \in X$ we can define the isotropy (stabilisor, fix group, "Standgruppe")

$$\Gamma_x := \{\gamma x = x\} \underset{\text{subgroup}}{\leq} \Gamma$$

Conversely, given any subgroup K of Γ, we have the K-fixed set (place, space)

$$X_K := \{x \in X \mid \gamma x = x, \text{ for all } \gamma \in K\} = \mathrm{Fix}(K)$$

Note: $x \in X_K \Leftrightarrow \Gamma_x \geq K$

We call ϱ a representation on X, Banach if all $\varrho(\gamma)$ are bounded linear: $X \to X$

For $X = \mathbb{R}^N$ we have (invertible) $N \times N$ matrices $\varrho(\gamma)$ and hence a group homomorphism

$\varrho : \Gamma \to GL(N)$

$\gamma \mapsto \varrho(\gamma)$

when we view groups Γ as matrix group, we automaticly mix up (identify) $\varrho(\gamma)$ with γ itself.

Example:

1. $\Gamma = (\mathbb{R}, +)$ acts on $X = \mathbb{R}^N$: flow
   $\Gamma_x = \mathbb{R}$ for equilibria
   $= p\,\mathbb{Z}$ for periodic orbits

2. Γ acts on $X = \Gamma$ by conjugation $\varrho(\gamma)g = \gamma g \gamma^{-1}$
   Then $\Gamma_g = \{\gamma \mid \gamma g = g\gamma\}$ = centralizor
   $X_K = \{g \in \Gamma \mid \gamma g = g \gamma\} = ?$

3. Γ acts on $X = \{\text{subgroups of } \Gamma\}$ by conjugation. Then $\Gamma_K = \{\gamma \mid \gamma K \gamma^{-1} = K\}$ = normalizor of K subgroup

Let $\mathcal{F}: W \to Z$, C^k, Banach, $\mathcal{F}(0) = 0$, $k \geq 1$.
Let ϱ_W, ϱ_Z be representations of the group Γ on W, Z. We call $\mathcal{F}$ equivariant w.r. to Γ if

$$\mathcal{F}(\varrho_W(\gamma)w) = \varrho_Z(\gamma)\mathcal{F}(w) \qquad \forall w \in W, \gamma \in \Gamma$$

short: $\mathcal{F}(\gamma w) = \gamma \mathcal{F}(w)$

[Vanderbauwhede] [Golubitsky & Stewart]
[Chossat & Lauterbach]

Theorem (equivariant LS reduction)

Assume $\mathcal{L} := \mathcal{F}'(0)$ is quasi-Fredholm with projections $P, Q = id - P$ in Z onto $(\text{Range}\,\mathcal{L})^{(\perp)}$ and projections $\tilde{P}, \tilde{Q} = id - \tilde{P}$ in W onto $(\ker \mathcal{L})^{(\perp)}$

Dyn 3 (before: $w = u \oplus v = (u,v)$; now: $u = \tilde{P}w$, $v = \tilde{Q}w$) 17.07.2011

Then $\mathcal{F}(u,v) = \sigma$, locally $\Leftrightarrow \begin{cases} P\mathcal{F}(u,v) = \sigma \Leftrightarrow v = V(u) \\ \text{and } \Phi(u) := Q\mathcal{F}(u,V(u)) = \sigma \end{cases}$

L-S-reduced equation; $\Phi, V \in C^k$, $V(\sigma) = \sigma$, $\Phi(\sigma) = \sigma$, $V'(\sigma) = \sigma$, $\Phi'(\sigma) = \sigma$

AND $\Phi : \tilde{P}W \to QZ$ are both equivariant

Remark: $P : Z \to Z$ equivariant $\Rightarrow$ Ker P, Range P are Γ-invariant. Indeed: $\underset{\text{Range } P}{\gamma P z} \overset{\text{equiv.}}{=\!=} \underset{\text{Range } P}{P \gamma z}$

Similarly $Pz = \sigma$

$\Rightarrow \sigma = \gamma P z = P \gamma z \Rightarrow \gamma z \in \text{Ker } P$. Therefore the restrictions of $\varrho_W(\Gamma)$, $\varrho_Z(\Gamma)$ provide representations on U, V, Range $\mathcal{L}$.

(Proof on page 44)

24.05.2011

continuous reps ϱ :	Γ topological group (i.e. composition is cont.) X = Banach, $\varrho(\gamma)$ bd. linear, $\gamma \mapsto \varrho(\gamma) \in L(X)$ cont.

(reps are assumed to be cont., wherever necessary)

Note: $X_K = \bigcap_{\gamma \in K} \text{Ker}(\varrho(\gamma) - id)$ = closed subspace $\Rightarrow$ also Banach

Equivariance:

$\mathcal{F} : W \to Z$ equivariant ("invariant", "covariant")

if $\mathcal{F}(\varrho_W(\gamma) w) = \varrho_Z(\gamma)\mathcal{F}(w)$ $(\forall w, \gamma)$

Rep.: Let $\mathcal{L} : W \to Z$ be linear and equivariant. Then Ker $\mathcal{L}$, Range $\mathcal{L}$ are Γ-invariant ie. $u \in \text{Ker } \mathcal{L}$ $\Rightarrow \varrho(\gamma) u \in \text{ker } \mathcal{L}$ $(\forall \gamma)$

Proof: $\mathcal{L}\varrho(\gamma) u = \varrho(\gamma)\mathcal{L} u = \sigma$ for $u \in \text{Ker } \mathcal{L}$

Similarly for Range □

Corollary: Eigenspaces and generalized eigenspaces of equivariant operators $\mathcal{L}$ are invariant

Proof: Consider $\mathcal{L} - \lambda \cdot id$, instead of $\mathcal{L}$.

Example: Consider $\dot{x} = f(x)$, $X = \mathbb{R}^N$, ode. Then equivariance $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$, group $x \in X$