Dyn 3 (before: $w = u \oplus v = (u,v)$; now: $u = \tilde{P}w$, $v = \tilde{Q}w$) 17.07.2011

Then $\mathcal{F}(u,v) = 0$, locally $\Leftrightarrow \begin{cases} P\mathcal{F}(u,v) = 0 \Leftrightarrow v = V(u) \\ \text{and } \Phi(u) := Q\mathcal{F}(u, V(u)) = 0 \end{cases}$

L-S-reduced equation; $\Phi, V \in C^k$, $V(0) = 0$, $\Phi(0) = 0$, $V'(0) = 0$, $\Phi'(0) = 0$

AND $\Phi: \tilde{P}W \to QZ$ are both equivariant

Remark: $P: Z \to Z$ equivariant $\Rightarrow$ Ker P, Range P are Γ-invariant. Indeed: $\underset{\text{Range } P}{\gamma Pz} \overset{\text{equiv.}}{=\!=} \underset{\text{Range } P}{P\gamma z}$

Similarly $Pz = 0$

$\Rightarrow 0 = \gamma Pz = P\gamma z \Rightarrow \gamma z \in \text{Ker } P$. Therefore the restrictions of $\varrho_W(\Gamma)$, $\varrho_Z(\Gamma)$ provide representations on u, v, Range $\mathcal{L}$.

(Proof on page 44)

24.05.2011

continuous reps ϱ: Γ topological group (i.e. composition is cont.) X = Banach, $\varrho(\gamma)$ bd. linear, $\gamma \mapsto \varrho(\gamma) \in L(X)$ cont.

7

(reps are assumed to be cont., wherever necessary)

Note: $X_K = \bigcap_{\gamma \in K} \text{Ker}(\varrho(\gamma) - id)$ = closed subspace $\Rightarrow$ also Banach

Equivariance:

$\mathcal{F}: W \to Z$ equivariant ("invariant", "covariant")

if $\mathcal{F}(\varrho_W(\gamma)w) = \varrho_Z(\gamma)\mathcal{F}(w)$ $\quad(\forall w, \gamma)$

Rep.: Let $\mathcal{L}: W \to Z$ be linear and equivariant. Then Ker $\mathcal{L}$, Range $\mathcal{L}$ are Γ-invariant ie. $u \in \text{Ker } \mathcal{L}$ $\Rightarrow \varrho(\gamma)u \in \text{ker } \mathcal{L}$ $\quad(\forall \gamma)$

Proof: $\mathcal{L}\varrho(\gamma)u = \varrho(\gamma)\mathcal{L}u = 0$ for $u \in \text{Ker } \mathcal{L}$

Similarly for Range $\square$

Corollary: Eigenspaces and generalized eigenspaces of equivariant operators $\mathcal{L}$ are invariant

Proof: Consider $\mathcal{L} - \lambda \cdot id$, instead of $\mathcal{L}$.

Example: Consider $\dot{x} = f(x)$, $X = \mathbb{R}^N$, ode. Then equivariance $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$, group $x \in X$

$\Leftrightarrow$ ($x(t)$ is a solution $\Longleftrightarrow$ $\gamma x(t)$ is a solution $(\forall \gamma)$)

If $f(x) = Ax$ linear, then equivariance simply means

$\gamma \cdot A = A \cdot \gamma \qquad (\forall \gamma \in \Gamma)$,

i.e. those matrices commute.

Equivariant LS:

Let $\mathcal{F}$ be as in $\boxed{7.1}$ (p. 23) and additionally assume equivariance of $\mathcal{F}$ AND equivariance of $\overset{(\sim)}{P}$ ($\Rightarrow \overset{(\sim)}{Q}$)

Then LS holds with equivariant maps $V: \operatorname{Ker} \mathcal{L} = \operatorname{Range} \tilde{P} \to \operatorname{Ker} \tilde{P} = \operatorname{Range} \tilde{Q}$

$\Phi : \operatorname{Ker} L = \ldots \to \operatorname{Range} Q$, complement to $\operatorname{Range} \mathcal{L}$

Moreover the lhs / rhs spaces are closed and Γ-invariant

Recall proof:

lhs / rhs Γ-inv. by Cor. applied to $\overset{(\sim)}{P}, \overset{(\sim)}{Q}$,

closed by quasi-Fredholm $\mathcal{L}$

Equivariance

$0 = P\mathcal{F}(u,v) \iff v = V(u)$

$0 = \gamma P \mathcal{F}(u,v) = P \gamma \mathcal{F}(\underbrace{Pw}_{w} + \underbrace{Qw}_{v})$

$= P\mathcal{F}(\gamma u, \gamma v) \iff \gamma v = V(\gamma u)$

$\overset{\shortparallel}{\gamma V(u)}$

Similarly for Φ $\square$

$\boxed{7.4}$ Equivariant stationary bifurcation

For all interesting cases $\dim \operatorname{Ker} D_x > 1$

C & R: ☹

Consider $0 = f(\lambda, x)$ with f as in section $\boxed{7.2}$

Trick: Consider subgroups $K \leq \Gamma$ and restrict

$f: \mathbb{R} \times X_K \to X_K$

Indeed, for $x \in X_K$, $\gamma \in K$:

$$\gamma f(\lambda, x) = f(\lambda, \gamma x) = \underbrace{f(\lambda, x)}_{x \in X_K}$$

Hence $f(\lambda, x) \in X_K$ flow invariant subgroups.

Theorem [Vanderbauwhede], [Cicogna] (1980s)

Assumptions as in C & R, section 2.2, but for $f_K := f|_{\mathbb{R} \times X_K} : \mathbb{R} \times X_K \to X_K$.

($\mathcal{L}_K := D_x f_K(0,0)$ Fredh., index $= 0$, $f(\lambda, 0) = 0$, $f \in C^k_{k \geq 2}$

(i) $\dim \ker \mathcal{L}_K = 1$

(ii) $D_\lambda D_x Q_K f_K(0,0) \neq 0$

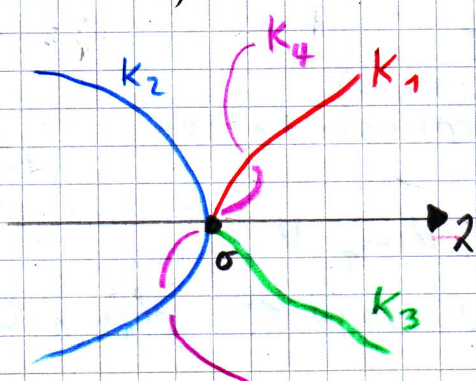
Then we obtain a local branch $(s, x(s))$ of nontrivial solutions

with $x(s) \in X_K$ i.e. $\Gamma_{x(s)} \geq K$)

and $\dot{x}(0) \neq 0$

Proof: Apply Crandall & Rabinowitz to f_K

Note: this may produce a mess (large number) of bifurcations, at the same bifurcation point $\lambda = 0$, $x = 0$



We have seen how kernels and eigenspaces come with group representations on them. The role of simple eigenvalues, 1d eigenspaces are played by irreducible representations

Def: We call a representation ϱ of Γ on a vectorspace U *irreducible*, if the only subspaces of U which are invariant under Γ, ϱ are the trivial supspace $\{0\}$, U.

(depends on whether U is $\mathbb{C}$- or $\mathbb{R}$-vector space)

Lemma [Issai Schur (1875–1941)]

Let ϱ be irreducible complex rep. of Γ on U.

Let $A: U \to U$ be linear & equivariant.

(i.e. A commutes with all $\varrho(\gamma)$, $\gamma \in \Gamma$)

Then there ex. $\lambda \in \mathbb{C}$ s.th. $A = \lambda \cdot id$ on U.

Proof: Let λ be any eigenvalue of A. Then

$\{0\} \neq \operatorname{Ker}(A - \lambda \cdot id) \leq U$ is invariant under Γ!

Because U is irred. we have $\operatorname{Ker}(A - \lambda \cdot id) = U$ $\square$

NOT TRUE for real reps., real λ.

Def.: A real rep. ϱ on U is called absolutely irreducible if Schur's Lemma holds with $\lambda \in \mathbb{R}$.

Example:

1. Let Γ be Abelian. Then all complex irred. reps of Γ are (complex) one dimensional.

   Proof: Pick $A = \varrho(\gamma_0)$, for any fixed $\gamma_0 \in \Gamma$, in Schur's Lemma.

   Hence $\tilde{\varrho}: \Gamma \to \mathbb{C}$

   $\gamma \mapsto \lambda$ (Schur)

   and this 1-d representation (e.g. given by the first component of $\varrho(\gamma)$) is just repeated $\dim_{\mathbb{C}} U$ times. Hence U irred $\Leftrightarrow$ $\dim_{\mathbb{C}} U = 1$ $\square$

2. Consider cont. complex irred. rept. ϱ of $\Gamma = SO(2)$ on U. Since Γ = Abelian, $\dim_{\mathbb{C}} U = 1$

   Moreover $\varrho(\underbrace{e^{it}}_{\in SO(2)}) = r(t)\, e^{i\varphi(t)}$

   with $\varrho(e^{i(t_1+t_2)}) = \varrho(e^{it_1})\varrho(e^{it_2})$, i.e.

$$\left.\begin{aligned} r(t_1+t_2) &= r(t_1)\cdot r(t_2) \\ \varphi(t_1+t_2) &= \varphi(t_1) + \varphi(t_2) \end{aligned}\right\} \text{cont}$$

$SO(2)$ compact $\Rightarrow r(t) \equiv 1$,

$\varphi : \mathbb{R}/_{2\pi\mathbb{Z}} \to \mathbb{R}/_{2\pi\mathbb{Z}}$, homomorphism

i.e. $\varrho(e^{it}) = e^{ikt}$ for some $k \in \mathbb{Z}$

Real irred: on $\mathbb{R}$, $k = 0$ on $\mathbb{R}^2$

$$\varrho(e^{it}) = \begin{pmatrix} \cos kt & -\sin kt \\ \sin kt & \cos kt \end{pmatrix} \qquad k \in \mathbb{Z} \setminus \{0\}$$

However k and $-k$ are equivalent

(i.e. ex. Γ-equivariant basis change mapping k to $-k$; reflection!

$k \neq 0$ not abs. irred.

③ $\Gamma = O(2) =$ rot. and refl. on $\mathbb{R}^2$, fixing zero

$= \{ e^{it}, \kappa e^{-it} \mid \kappa z = \bar{z} \}$

reflection on x-axis, NOT complex linear

has real irred representations on $\mathbb{R}$:

$1 \qquad (\varrho(\gamma) = 1)$

$-1 \qquad (\varrho(\text{rot}) = 1$

$\qquad \varrho(\text{reflection}) = -1)$

on $\mathbb{C}$: $\varrho(e^{it}) z = e^{ikt} z$, some $k \in \mathbb{N}$

$\varrho(\kappa) z = \bar{z}$

complex irred:

on $\mathbb{C}^2$: $\varrho(e^{it}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{ikt} z_1 \\ e^{-ikt} z_2 \end{pmatrix}$

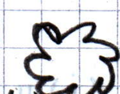
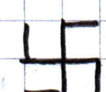
$\varrho(\kappa) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}$

(recall $\kappa e^{it} \kappa = e^{-it}$)

④ cyclic $\mathbb{Z}_n = \langle e^{2\pi i/n} \rangle = [\{ e^{2\pi i j/n}, j = 0, \dots, n-1 \}]$ $j \pmod n$

dihedral – $D_n = \langle e^{2\pi i/n}, \kappa \rangle$

e.g. D_n = symmetry of regular n-gon  

$\mathbb{Z}_n$ = symmetry of  , …

complex irred: $\mathbb{Z}_n = \varrho(e^{2\pi i/n}) = e^{2\pi i k/n}$ NOT abs. red.

real irred: $\mathbb{Z}_n$ dito, but k equivalent to $-k$

D_n: take account of k as before abs irred $n \geq 3$

⑤ $SO(3)$, $O(3)$

Representation: $\varrho(g)h(x) := h(g^{-1}x)$

on scalar functions $h: \mathbb{R}^3 \to \mathbb{R}$ (or $\mathbb{C}$)

$h(rx) = r^{\ell} h(x) \quad \forall r \in \mathbb{R}$

(ℓ-Irred reps. on $U_\ell := \{ h \mid h$ homogeneous polynomial of degree $= \ell$, $\Delta h = 0 \}$

$\dim_{\mathbb{R}} U_\ell = 2\ell + 1$. In polar coord. $r=1$, ϑ, φ on $S^2 \subset \mathbb{R}^3$, the space U_ℓ is spanned by Re / Im of $Y_{\ell m}(\vartheta, \varphi) := P_\ell(\cos\vartheta) e^{im\varphi}$, $m = 0, \dots, \ell$

where P_ℓ are the Legendre polynomials (orthogonalize $1, x, x^2, \dots$ by L^2 scalar product $\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx$

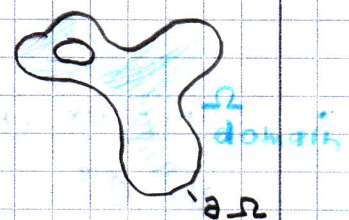
$Y_{\ell m}$: spherical harmonics. abs irred.

$O(3) = SO(3) \cup k\, SO(3)$, $k = -id$

$\varrho(-id) = (-1)^\ell$ on U_ℓ

Excursion (Laplace operator Δ)

Consider bd. open subset $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary. Laplace



$$(\Delta u)(x) := \sum_{i=1}^{n} \partial_{x_i}^2 u(x)$$

ⓐ Poisson equation

$$\Delta u(x) = f(x)$$
$$u(x) = 0 \quad \text{on } \partial\Omega$$

"Solution": $u = \Delta^{-1} f$

Fact1: $\Delta: C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega}) \cap \{b.c\} \to C^{0,\alpha}(\Omega)$

[Schauder] is bounded, linear, bij., hence bd. inverse

$$C^{0,\alpha} = \{ u \in C^0 \mid \|u\|_{C^{0,\alpha}} < \infty \}, \qquad 0 \leq \alpha \leq 1$$

$$\|u\|_{C^{0,\alpha}} = \|u\|_{C^0} + \sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{u(x) - u(y)}{|x-y|}$$

$$\|u\|_{C^{k,\alpha}} := \sum_{j=0}^{k} \|u^{(j)}\|_{C^{0,\alpha}}$$

 $\Delta^{-1}: L^p \to W^{2,p} = \{u \mid u'' \in L^p\} \cap \{b.c.\}$ | $\|u\|_{W^{2,p}} = \|u\|_{L^p} + \|u'\|_{L^p} + \|u''\|_{L^p}$

Fact 2: $\Delta: H^2(\Omega) \cap \{b.c\} \to L^2(\Omega)$

[Sobolev] is bd., lin., bij., hence bd. inverse

$H^2(\Omega) = \{u \mid \|u\|_{H^2} < \infty\}$

here $\|u\|_{H^k} := \sum_{j=0}^{k} \|u^{(j)}\|_{L^2}$

(either: completion of $C^\infty(\Omega) \cap \{b.c\}$ w.r. to this norm, OR: u with weak derivatives in L^2 up to order k (defined via integration by parts)

$(\int u'' g = \int u g'')$

[Evans]

(b) spectral theory

Consider $L := id + \lambda \Delta^{-1}$ on $X = L^2$ or $C^{0,\alpha}$.

Because Δ^{-1} is compact

(Arzela-Ascoli type for $C^{2,\alpha} \overset{cpt}{\hookrightarrow} C^{0,\alpha}$, resp. Sobolev embedding for $H^2 \hookrightarrow L^2$),

the operator L is Fredholm.

kernel $L \neq \{0\}$ for $\lambda = \lambda_0 < \lambda_1 < \dots$, where $\frac{1}{\lambda_k}$ are the eigenvalues of the inverse $-\Delta^{-1}$.

The corresponding eigenspaces are algebraic = geometric and we can decompose

$$X = \text{Ker}(id + \lambda_k \Delta^{-1}) \underset{\perp_{L^2}}{\oplus} \text{Range}(id + \lambda_k \Delta^{-1})$$

where Range $\perp$ ker. This reflects the fact that Δ is self-adjoint w.r. to the scalar product $\langle f, g \rangle := \int_\Omega f(x) g(x)\, dx$

We also call λ_k the eigenvalues of $-\Delta$

The dec. of X defines orthogonal eigenprojections.

(c) symmetry

For symmetric domains $\gamma\Omega = \Omega$ (as a set)

for $\gamma \in \Gamma \leq O(n)$, the operator Δ^{-1}

becomes equivariant w.r. to the following representation of Γ on functions,

$$(\varrho(\gamma)u)(x) := u\,(\gamma^{-1}x)$$

$\forall \gamma \in \Gamma,\ u \in X$ — usual matrix in $\Gamma \subseteq O(n)$

$$\varrho(\gamma_1)\varrho(\gamma_2) = \varrho(\gamma_1\gamma_2)$$

17.04.2011

Nachtrag:

(to p. 37)

Proof: Equivariance of V

$$\gamma P \mathcal{F}(u, \gamma^{-1} V(\gamma u)) = P\mathcal{F}(\gamma u, V(\gamma u)) = 0$$

$$\Rightarrow P\mathcal{F}(u, \gamma^{-1}V(\gamma u)) = 0$$

$$\xRightarrow[V(u)\text{ unique}]{} \gamma^{-1} V(\gamma u) = V(u)$$

$$\gamma\,\Phi(u) = \gamma Q \mathcal{F}(u, V(u)) = Q\mathcal{F}(\gamma u, \gamma V(u))$$

$$= Q\mathcal{F}(\gamma u, V(\gamma u)) = \Phi\gamma(u) \qquad \square$$

$\varrho(\gamma^{-1}) = (\varrho(\gamma))^{-1}$
$u = \hat{P}w,\ v = \tilde{Q}w$

31.03.2011

8 In fact Δ^{-1} (and Δ) are self adjoint:

$$\langle u, \Delta^{-1} v\rangle_{L^2(\Omega)} = \langle \Delta^{-1} u, v\rangle_{L^2(\Omega)} \qquad (\forall u, v \in \mathcal{D}(\Delta) = (H^2, W^{2,p}, C^{2,\alpha}) \cap \{b.c\}$$

In particular $\ker(\lambda + \Delta) = (\text{Range}(\lambda+\Delta))^{\perp_{L^2}}$ $\lambda \in \mathbb{R}$!

indeed $\langle \underset{\text{ker}}{u}, \underset{\text{Range}}{(\lambda+\Delta)v}\rangle = \langle(\lambda+\Delta)u, v\rangle = 0$

$$u \in \ker(\lambda+\Delta) \Rightarrow u \in (\text{Range}(\lambda+\Delta))^\perp$$

$$\dim \ker \underset{\substack{\text{Fredh.}\\ \text{index}=0\\ \text{for } id + \lambda\Delta^{-1}}}{=} \dim(\text{Range}(\lambda+\Delta)^\perp) \Rightarrow \ker = \text{Range}^\perp$$

Projections $(\tilde{P})$, $(\tilde{Q})$ can be chosen orthogonal!

Note $\varrho_X(\gamma)$

1. $\langle \gamma u, \gamma v\rangle_{L^2(\Omega)} = \langle u, v\rangle_{L^2(\Omega)}$ $\qquad (\forall u, v \in L^2(\Omega), \gamma \in \Gamma)$:
i.e. the $\varrho_X(\sigma)$ act orthogonally.