becomes equivariant w.r.t. to the following representation of Γ on functions,

$$(\varrho(\gamma)u)(x) := u\ (\gamma^{-1}x)$$

$\forall \gamma \in \Gamma,\ u \in X$ usual matrix in $\Gamma \in O(n)$

$$\varrho(\gamma_1)\varrho(\gamma_2) = \varrho(\gamma_1\gamma_2)$$

Nachtrag: 17.04.2011

(to p. 37)

Proof: Equivariance of V

$$\gamma P \mathcal{F}(u, \gamma^{-1} V(\gamma u))$$
$$= P\mathcal{F}(\gamma u, V(\gamma u)) = 0$$
$$\Rightarrow P\mathcal{F}(u, \gamma^{-1} V(\gamma u)) = 0$$
$$\underset{V(u)\text{ unique}}{\Longrightarrow} \gamma^{-1} V(\gamma u) = V(u)$$

$$\gamma\, \Phi(u) = \gamma\, Q\, \mathcal{F}(u, V(u)) = Q\mathcal{F}(\gamma u, \gamma V(u))$$

$\varrho(\gamma^{-1}) = (\varrho(\gamma))^{-1}$

$u = \tilde{P}w,\ v = \tilde{Q}w$

$$= Q\mathcal{F}(\gamma u, V(\gamma u)) = \Phi\gamma(u) \qquad \square$$

31.03.2011

8 In fact Δ^{-1} (and Δ) are self adjoint:

$$\langle u, \Delta^{-1} v\rangle_{L^2(\Omega)} = \langle \Delta^{-1} u, v\rangle_{L^2(\Omega)} \qquad (\forall u, v \in \mathcal{D}(\Delta) = (H^2, W^{2,p}, C^{2,\alpha}) \cap \text{b.c.}$$

$\cap \{b.c\}$

In particular $\quad \operatorname{Ker}(\lambda + \Delta) = (\operatorname{Range}(\lambda+\Delta))^{\perp_{L^2}}$ $\quad \lambda \in \mathbb{R}!$

indeed $\quad \langle \underset{\substack{\cap \\ \text{Ker}}}{u}, \underset{\substack{\cap \\ \text{Range}}}{(\lambda+\Delta) v}\rangle = \langle (\lambda+\Delta) u, v\rangle = 0$

$u \in \operatorname{Ker}(\lambda+\Delta) \Rightarrow u \in (\operatorname{Range}(\lambda+\Delta))^{\perp}$

$\dim \operatorname{Ker} \underset{\substack{\text{Fredh.} \\ \text{index}=0 \\ \text{for } id + \lambda\Delta^{-1}}}{=} \dim(\operatorname{Range}(\lambda+\Delta)^{\perp})$

$\Rightarrow \operatorname{Ker} = \operatorname{Range}^{\perp}$

Projections $(\tilde{P})$, $(\tilde{Q})$ can be chosen orthogonal!

Note $\varrho_X(\gamma)$

1. $\langle \gamma u, \gamma v\rangle_{L^2(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} \qquad (\forall u, v \in L^2(\Omega), \gamma \in \Gamma)$:

i.e. the $\varrho_X(\sigma)$ act orthogonally.

$$\langle \gamma u, \gamma v\rangle_{L^2(\Omega)} = \int (\rho(\gamma)u)(x)\cdot(\rho(\gamma)v)(x)\,dx$$

$$= \int_\Omega u(\underbrace{\gamma^{-1}x}_{y})\cdot v(\underbrace{\gamma^{-1}x}_{y})\,\underbrace{dx}_{\underbrace{\det\gamma}_{1}\cdot dy}$$

$$= \int_{\gamma\Omega=\Omega} u(y)v(y)\,dy = \langle u,v\rangle.$$

2. Orthogonal projections onto Γ-invariant subspaces are Γ-equivariant $\quad \gamma u = \underbrace{\gamma v}_{\in V} \oplus \underbrace{\gamma w}_{\in V^\perp}$ $\in V^\perp$

$\langle v,w\rangle = \langle \gamma v, \gamma w\rangle$

3. $\Delta^{-1}: L^p \to L^p$ is Γ-equivariant

$\Delta: W^{2,p} \to L^p$ is Γ-equivariant

$\Delta: C^{2,k} \to C^{0,k}$ is Γ-equivariant

indeed:

$$(\gamma\Delta)(x) = (\Delta u)(\gamma^{-1}x) = tr(u''(\gamma^{-1}x))$$

$$(\Delta(\gamma u))(x) = \Delta_x u(\gamma^{-1}x) \underset{\text{chain rule}}{=} tr\big((\cancel{\gamma^{-1}})^T(u''(\gamma^{-1}x))\cancel{\gamma^{-1}}\big)$$

4. Eigenspaces

$$\ker(\lambda+\Delta) \underset{\lambda\neq 0}{=} \ker(\overbrace{id + \tfrac{1}{\lambda}\Delta^{-1}}^{L}) \text{ is } \Gamma\text{-invariant}$$

because L is equivariant.

5. Because Δ^{-1} is self-adjoint, the eigenprojections, according to the orthogonal splitting $X = \ker L \oplus \operatorname{Range} L$ are orthogonal and Γ-equivariant.

Example: Consider $\Delta u + \lambda h(u) = 0$ on $\Omega \subset \mathbb{R}^{2,3}$ (open bd.)

$\boxed{\text{functional } J(u) := \int_\Omega \tfrac12(\nabla u)^2 - \lambda H(u)\,dx}$ $\quad u = 0$ on $\partial\Omega \in Lip$

$h: \mathbb{R}\to\mathbb{R}$, C^k, $k\geq 2$, $h(0)=0$, $h'(0)=1$

(e.g. Euler rod, $\Omega = (0,1)$

Try Crandall & Rabinowitz; trick:

$f(\lambda,u) := u(x) + \lambda\,\Delta^{-1} h(u(x))$

$f: H^2 \cap \{b.c\} \to H^2 \cap \{b.c\} = X$ (or $X = C^{0,\alpha}$)

$h \in C^k \Rightarrow f \in C^k$ (because $H^2 \hookrightarrow C^0$)

Linearization: $D_u f(\lambda, 0) = id + \underbrace{\lambda\,\Delta^{-1}}_{\text{cpt., Sobolev}}$, Fredholm index $= 0$

Also Γ-equivariance for

$\Gamma = \{\gamma \in O(n) \mid \gamma\Omega = \Omega\}$.

id, Δ^{-1} : Ok $\quad (S(\gamma) h(u(x)))(x) = h(u(\gamma^{-1}x))$

$= (\tilde{h}(S(\gamma)u))(x)$

This smells like Crandall & Rabinowitz:

$\boxed{(\tilde{h}(u(\cdot))(x) := h(u(x))}$

Check their condition (ii):

$Q\, D_\lambda D_u f(\lambda = 0, u = 0)\Big|_{\ker(id + \lambda\Delta^{-1})} = Q\Delta^{-1}\Big|_{\ker(id+\lambda\Delta^{-1})}$

$= -\frac{1}{\lambda}\, \underbrace{Q}_{id}\Big|_{\ker(id+\lambda\Delta^{-1})} = -\frac{1}{\lambda}\Big|_{\ker(id+\lambda\Delta^{-1})} \neq 0$

Equivariance: orthogonal eigenprojection is Γ-equivariant (see remarks (p.)) together with Γ-equivariance of f, we may thus apply C & R on $X_K = \text{Fix}(K)$, for any subgroup K of Γ.

So we only have to check condition (i)

(i) $\dim \ker(id + \lambda\Delta^{-1})|_{X_K} = 1$

Frequently it turns out that this eigenspace of $-\lambda$ for Δ is irreducible under the group action $S_X(\gamma)$.

Example: $\Omega = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\vartheta \in \Gamma = S^1 = SO(2)$

$(\gamma_\vartheta(\vartheta)\, u)(x) := u(x - \vartheta) \quad \in L^2 = X$

Fourier: $u(x) = \sum_{k\in\mathbb{Z}} \underbrace{u_k\, e^{ikx}}_{\text{proj onto irred. component}}$

eigenvalue: $+k^2$ for $-\Delta = -\partial_x^2$

Example 2: $\Omega = S^2$, $\Gamma = SO(3)$

eigenfunction $Y_{\ell m}(\vartheta, \varphi) = P_\ell(\cos\vartheta) e^{im\varphi}$,

$$-\Delta Y_{\ell m} = \ell(\ell+1) Y_{\ell m}$$

$\Gamma = SO(3)$ on S^2:

$$L^2 = \bigoplus_{\ell=0}^{\infty} V_\ell \qquad V_\ell = \langle Y_{\ell m} \mid m = -\ell, \dots, +\ell \rangle$$

$$\dim V_\ell = 2\ell + 1$$

With group actions of Γ the irreducible representations of Γ play the same role as the eigenspaces of simple eigenvalues, in absence of Γ. In particular $\dim \operatorname{Ker}(id + \lambda\Delta^{-1}) = 1$ is rarely to be expected in presence of interesting groups Γ ("interesting" :⇔ Γ as irred reps of dim ≥ 2)

and C & R fail miserably. But Vanderbauwhede / Cicogna save our day, if

$$\dim \operatorname{Ker}(id + \lambda\Delta^{-1})|_{X_K} = 1$$

<u>Purely algebraic question:</u> for any finite-dim irred. rep. of Γ find subgroups $K \leq \Gamma$ such that $\dim \operatorname{Fix}(K) = 1$

Usually people choose for $K \leq \Gamma$ a maximal isotropy subgroup, i.e.

$u \in \operatorname{Fix}(K) \Rightarrow \Gamma_u = K$ or Γ where in any nontrivial repr.

nothing in between

$\Gamma_u = \Gamma \Rightarrow u = 0$

See [Chossat & Lauterbach] which also corrects MANY errors of previous authors, e.g. [Golubitsky & Stewart], ...

7.5 Hopf bifurcation

We seek small nontrivial T-periodic solutions

$$\dot{\hat{x}}(t) = h(\lambda, \hat{x}(t)) \qquad \hat{x} \in \mathbb{R}^N$$

for $h(\lambda, 0) = 0$, $h \in C^k$, $k \geq 2$ $\quad A(\lambda) := D_{\hat{x}} h(\lambda, 0)$

Linear $\quad \dot{y} = A(\lambda) y$

has 2π-periodic nontrivial solutions $\Leftrightarrow$

$$\pm n i \in \operatorname{spec} A(\lambda) \quad \text{for some } n \in \mathbb{N} \setminus \{0\}.$$

period 2π is not necessarily minimal here.

A nonlinear bifurcation version of this simple observation was initiated by Poincaré (shape of rotation fluid drops, wz stars)

[Andronov] (1920s), [Eberhard Hopf (1902-1983)] (1942), in $\mathbb{R}^n$. We will study the Hopf version, as formulated by Crandall & Rabinowitz (1976), with a proof by Vanderbauwhede (1980s) (Pitman back)

Theorem [Hopf] bifurcation Let h be as above and assume

(i) (imaginary eigenvalue) $\pm i \in \operatorname{spec} A(0)$ is an algebraicly simple eigenvalue

(ii) (nonresonance) $\pm n i \notin \operatorname{spec} A(0)$ for $n \in \mathbb{N}$, $n \geq 2$

(iii) (transverse crossing) the continuation

$\mu(\lambda) \in \operatorname{spec} A(\lambda)$ of $\mu(0) = \pm i$ crosses the imaginary axis transversely, at $\lambda = 0$: $\left.\frac{d}{d\lambda}\right|_{\lambda=0} \operatorname{Re} \mu(\lambda) \neq 0$.

Then there ex. a bifurcating branch

$$s \mapsto (\lambda(s), T(s), \hat{x}(t, s))$$ of solutions

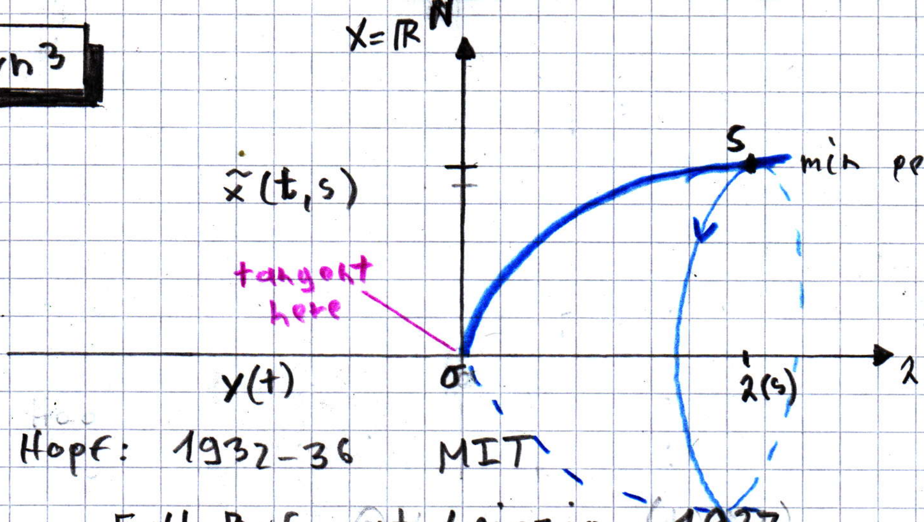
$\hat{x}(\cdot, s)$ of ⊛ such that $T(s)$ = minimal period of $t \mapsto \hat{x}(t, s)$ for $s > 0$, and $\lambda(0) = 0$,

$T(0) = 2\pi$, $\hat{x}(t, 0) = 0$, and

$$y(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} \hat{x}(t, 0) \in \operatorname{Eig}(\pm i, A(0)) \setminus \{0\}$$

Dyn 3

31.05.2011



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