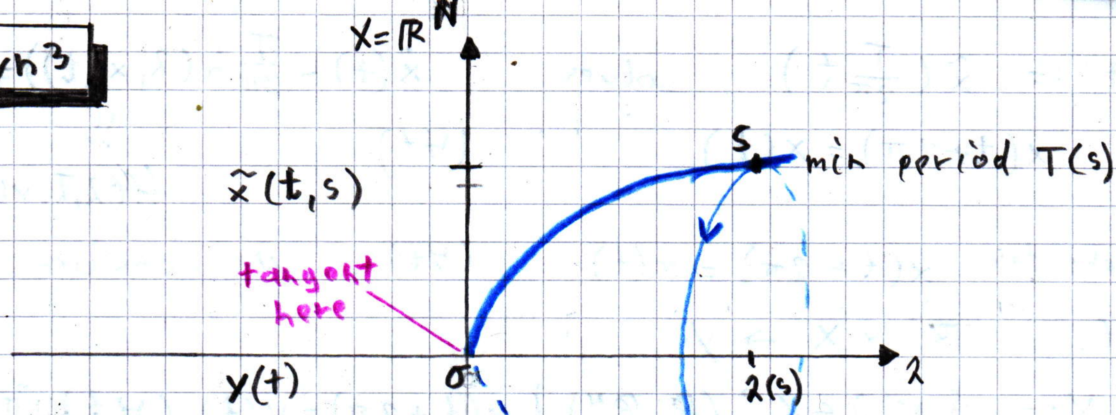

Hopf: 1932–36 MIT

Full Prof. at Leipzig (1937)

Institut für Segelflug 1942

Full Prof. at München 1944

Full Prof. at Indiana 1947

math genealogy

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9 Proof uses equivariant Lyapunov-Schmidt reduction:

$\mathcal{F}(x) = 0, \quad \mathcal{L} = \mathcal{F}'(0)$ Fredholm.

Projections P onto Range $\mathcal{L}$, $\tilde{P}$ onto $\ker \mathcal{L}$,

$\overset{(\sim)}{Q} := \mathrm{id} - \overset{(\sim)}{P}$; $\quad u = \tilde{P}x, \quad v = \tilde{Q}x.$

Reduction:

$$\mathcal{F}(x) = 0 \iff \begin{cases} P\mathcal{F}(u,v) = 0 \overset{}{\underset{\text{IFT}}{\iff}} v = V(u) \\ Q\mathcal{F}(u,v) = 0 \end{cases}$$

$$\iff \Phi(u) := Q\mathcal{F}(u, V(u))$$

For Γ-equivariant $\mathcal{F}$, P, $\tilde{P}$, the LS-reduced function Φ is also equivariant.

Idea of Proof: [Vanderbauwhede]

1. equivariant functional analysis set-up
2. perform equivariant LS-reduction
3. discuss (and solve) L-S reduced equation $\Phi = 0$
4. check this actually proves theorem

Proof: Step 1: Functional Analysis setup

suppose $\tilde{x}(t+T) = \tilde{x}(t)$ has period $T > 0$

Then $\tilde{x}(t)$ solves $*$, iff

$x(t) := \tilde{x}\left(\frac{T}{2\pi}t\right)$ solves $\underbrace{\dot{x}(t) - \frac{T}{2\pi} h(\lambda, x(t))}_{(f(\lambda,T,x(\cdot)))(t)} \overset{!!}{=} 0$

and $x(t+2\pi) = x(t)$ $(\forall t)$

We consider $x(t+2\pi) = x(t)$ $(\forall t)$. We consider

$$f: \quad \mathbb{R}^2 \times X \to Y$$

where $Y := \{ y(\cdot) \in C^0(\mathbb{R}, \mathbb{R}^N) \mid y(t+2\pi) = y(t) \ (\forall t) \}$ sup-norm

$X := Y \cap C^1$, sup norm of $y, \dot{y}$, Banach spaces.

Linearization:

$$L(\lambda, T)y = D_x f(\lambda, T, 0)y = \dot{y} - \frac{T}{2\pi} \underbrace{A(\lambda)}_{D_x h(\lambda,0)} y$$

$$\mathcal{L} := L(0, 2\pi)$$

All $L(\lambda, T)$ are Fredholm, because $\left.\begin{array}{l} y \mapsto -\frac{T}{2\pi} A(\lambda) y \\ \tilde{C}^1 : X \to Y = \tilde{C}^0 \end{array}\right\}$ compact Arzela Ascoli

and $\frac{d}{dt}: y \mapsto \dot{y}$
$X = \tilde{C}^1 \to \tilde{C}^0 = Y$

Indeed: $\ker\left(\frac{d}{dt}\right) = \{\text{const.}\} \cong \mathbb{R}^N$ and $\text{Range}\left(\frac{d}{dt}\right) = \{\text{const.}\}^\perp$ in $Y = \tilde{C}^0$

(direct proof of Fredholm: Fourier series).

$\dot{y}(t) = g(t)$
$\Leftrightarrow y(t) = y(0) + \int_0^t g(\tau)\, d\tau$ $\quad 2\pi$-per.
$\Leftrightarrow \int_0^{2\pi} \vec{e}_k \, g(\tau)\, d\tau = 0$
($\forall$ unit vectors $\vec{e}_k \in \mathbb{R}^N$)

Step 2: L-S-reduction

$y \in \ker \mathcal{L} \Leftrightarrow \dot{y} - A(0)y = 0$ and y is 2π-periodic

Fourier: $y = \sum y_k e^{ikt}$; $0 = \dot{y} - A(0)y$
$$= \sum (ik - A(0)) y_k e^{ikt}$$

$$\Leftrightarrow (ik - A(0)) y_k = 0 \quad (\forall k \in \mathbb{Z})$$

$$\underset{(i)}{\Leftrightarrow} y_k = 0, \ k \neq \pm 1 \quad \text{and} \quad y_{\pm 1} \in \text{Eig}(\pm i, A(0))$$

Abusing complex notation for real eigenspace, we can write

$\ker \mathcal{L}$: spanned by $e^{it} y_r \neq 0$, $i y_r = A(0) y_r$

Let $(Qy)(t) := \frac{1}{2\pi} \int_0^{2\pi} \underbrace{e^{it}\, \bar{y}_\ell^T}_{e_\ell} y(\tau)\, d\tau \cdot \underbrace{e^{it} y_r}_{e_r}$, where

$-i y_\ell^T = y_\ell^T A(0)$ and $\bar{y}_\ell^T y_r = 1$

$\underset{\text{short hand}}{=} \langle e_\ell, y\rangle\, e_r$, $\qquad \langle g_1, g_2\rangle := \frac{1}{2\pi}\int_0^{2\pi} \bar g_1(t)^T g_2(t)\,dt$;

scalar product

(i) $\ker Q = \operatorname{Range}\mathcal{L}$, $\operatorname{Range} Q = \ker\mathcal{L}$

Eigenprojection onto $\operatorname{Eig}(i, A(0))$ written as $y_r \bar y_\ell^T$

check: $Q\underbrace{\mathcal{L}y}_{\operatorname{Range}\mathcal{L}} = \langle e_\ell, \mathcal{L}y\rangle \underbrace{e_r}_{\in\ker\mathcal{L}}$

$$\langle e_\ell, \mathcal{L}_y\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{it}\, \bar y_\ell^T \underbrace{(\dot y(\tau) - A(0)y(\tau))}_{\mathcal{L}y(\tau)}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Big(\big(\underbrace{-\tfrac{d}{dt}\, e^{i\tau}}_{+i}\big)\, \bar y_\ell^T y(\tau) - e^{-i\tau}\, i\, \bar y_\ell^T y(\tau)\Big)d\tau = 0$$

$\Rightarrow \operatorname{Range}\mathcal{L}^{+i} \subseteq \ker Q$. But $\dim_{\mathbb{C}}\ker\mathcal{L} = 1 \underset{\substack{\text{Fredh.}\\ \text{index}\\ =0}}{\Rightarrow} \operatorname{codim}_{\mathbb{C}}\operatorname{Range}\mathcal{L} = 1$

i.e. $\operatorname{Range}\mathcal{L} \perp e_\ell$ $\longrightarrow$ $\operatorname{Range}\mathcal{L} = (e_\ell)^\perp = \ker Q$

(ii) $Q^2 = Q$, i.e. $Qe_r = e_r$

check: $Qe_r = \langle e_\ell, e_r\rangle\, e_r$

$\langle e_\ell, e_r\rangle = \cancel{\frac{1}{2\pi}\int_0^{2\pi} e^{-i\tau}\, \bar y_\ell^T}$

$\underbrace{e^{i\tau} y_r}_{e_r} = 1$

$$\int_0^{2\pi} e^{-i\tau}\, \bar y_\ell^T \dot y(\tau)\,d\tau$$

$$= \underbrace{\left[e^{-i\tau}\, y_\ell^T y(\tau)\right]_0^{2\pi}}_{\text{periodic}} - \int_0^{2\pi}\frac{d}{dt} e^{i\tau}\, \bar y_\ell^T y(\tau)\,d\tau$$

$$= 0 + i\int_0^{2\pi} e^{i\tau}\, \bar y_\ell^T y(\tau)\,d\tau$$

(iii) equivariance of Q: $\vartheta Q y = Q\vartheta y$, where

$\vartheta \in \Gamma = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ $\qquad (\vartheta y)(t) := y(t+\vartheta)$

$(\vartheta Q y)(t) = \langle e_\ell, y\rangle\, e^{i(t+\vartheta)} y_r$

$(Q\vartheta y)(t) = \left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i\tau}\, \bar y_\ell^T y(\tau+\vartheta)\,d\tau\right)\cdot e^{it} y_r$

$$\underset{2\pi\text{-per}}{=} \left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(\hat\tau-\vartheta)}\, \bar y_\ell^T y(\hat\tau)\,d\hat\tau\right) e^{it} y_r$$

$$= \langle e_\ell, y\rangle\, e^{i(t+\vartheta)}\, y_r = (\vartheta Q y)(t)$$

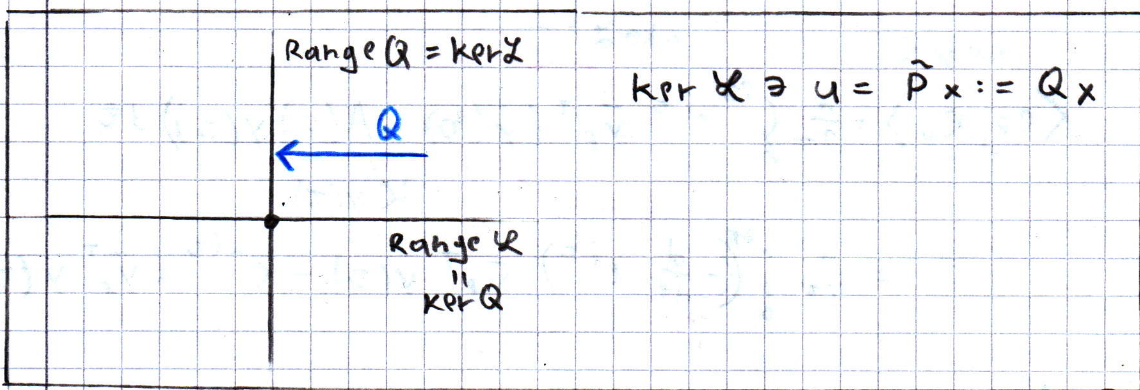
$$\underline{\text{Step } 0:} \quad f(\lambda, T, \vartheta x) = \vartheta f(\lambda, T, x)$$

$$\underline{\text{check:}} \quad \vartheta f(\lambda, T, x(\cdot)) = \dot{x}(t+\vartheta) - \frac{T}{2\pi} h(\lambda, x(t+\vartheta))$$

$$f(\lambda, T, (\vartheta x)(\cdot))(t) = \frac{d}{dt}((\vartheta x)(t)) - \frac{T}{2\pi} h(\lambda, (\vartheta x)(t))$$

$$= \dot{x}(t-\vartheta) - \frac{T}{2\pi} h(\lambda, x(t+\vartheta))$$

$\Rightarrow$ L-S with $Q := Q$



$\underline{\text{Step 3:}}$ L-S-reduction now produces LS-reduced equation:

$$\Phi(\lambda, T, \underset{\mathbb{C}}{\hat{z}}) := Q f(\lambda, T, \underbrace{\overset{u}{\overset{\downarrow}{z e^{it}}}\, y_r}_{\ker \mathcal{L}} + V(\lambda, T, z e^{it} y_r))$$

$$= \underbrace{\langle e_\ell, f(\lambda, T, z e^{it} y_r + V(\lambda, T, z e^{it} y_r)) \rangle}_{\tilde{\Phi}(\lambda, T, z) \in \mathbb{C}} e_r \overset{!}{=} 0$$

inherits equivariance:

$$\boxed{\tilde{\Phi}(\lambda, T, e^{i\vartheta} z) = e^{i\vartheta} \tilde{\Phi}(\lambda, T, z)}$$

$$\vartheta(z e^{it} y_r) = z e^{i(t+\vartheta)} y_r = e^{i\vartheta} z \underbrace{e^{it} y_r}_{e_r}$$

$\underline{\text{Note:}}$ (i) $\tilde{\Phi}(\lambda, T, 0) = 0$ (by equivariance, or directly: $f(\lambda, T, 0) = 0$, $V(\lambda, T, 0) = 0$)

(ii) $\varphi(\lambda, T, \hat{z}) := z^{-1} \tilde{\Phi}(\lambda, T, z)$ is M-inv:

$$\varphi(\lambda, T, e^{i\vartheta} z) = \varphi(\lambda, T, z) = \varphi(\lambda, T, \overset{=: r}{|z|})$$

$$\varphi(\lambda, T, 0) := \lim_{z \to 0} \frac{1}{z} \tilde{\Phi}(\lambda, T, z) = \lim_{z \to 0} \cancel{\frac{1}{z}} \frac{1}{2\pi} \int_0^{2\pi} e^{-it}\, \bar{y}_\ell^T D_x f(\lambda, T, 0)\, \cancel{z} e^{it} y_r \, dt$$

$Q D_x V(\lambda, T, 0) = 0$

$$= \cancel{\frac{1}{2\pi}} \cancel{\int_0^{2\pi}} \cancel{e^{-it}}\, \bar{y}_\ell^T \left(\cancel{\frac{d}{dt}}\, i - \frac{T}{2\pi} A(\lambda) \right) \cancel{e^{it}}\, y_r \, dt$$

$= \bar{y}_\ell^T \left(i - \frac{T}{2\pi} A(\lambda)\right) y_r$ and $\varphi \in C^{k-1}$

Now study $\varphi(\lambda, T, r) = 0$

Claim: (iv) $\det D_{(\lambda,T)} \begin{pmatrix} \operatorname{Re} \varphi(\lambda,T,0) \\ \operatorname{Im} \varphi(\lambda,T,0) \end{pmatrix} \neq 0 \overset{\text{IFT}}{\Rightarrow} \begin{cases} \varphi(\lambda,T,r) = 0 \\ \Leftrightarrow (\lambda,T) = (\lambda,T)(r) \end{cases}$

(iii) $\varphi(0, 2\pi, 0) = 0$

$\lambda(0) = 0$
$T(0) = 2\pi$

(iii): $\varphi(0, 2\pi, 0) = \bar{y}_\ell^T (i - A(0)) y_r = 0$

(iv) Let $y(\lambda)$ = eigenvector of eigenvalue $\mu(\lambda)$ of $A(\lambda)$, $\mu(0) = +i$.

$\mu(\lambda) y(\lambda) = A(\lambda) y(\lambda)$; $\quad ' := \frac{d}{d\lambda}$

$\mu' y + \mu y' = A y' + A' y$

at $y = 0$: $\bar{y}_\ell^T \mu' y_r = \underbrace{\bar{y}_\ell^T (-i + A)}_{=0} y' + \bar{y}_\ell^T A' y_r$

i.e.: $\mu' = \bar{y}_\ell^T A'(0) y_r$

This implies $\partial_\lambda \varphi(\lambda = 0, T = 2\pi, r = 0) = -\bar{y}_\ell^T \overset{=1}{\frac{T}{2\pi}} A'(0) y_r = -\mu'(0)$

$\partial_T \varphi(\lambda = 0, T = 2\pi, r = 0) = \bar{y}_\ell^T \left(-\frac{1}{2\pi} A(0)\right) y_r = -\frac{i}{2\pi} \underbrace{\bar{y}_\ell^T y_r}_{=1}$

$\Rightarrow \det \partial_{(\lambda,T)} \begin{pmatrix} \operatorname{Re} \varphi \\ \operatorname{Im} \varphi \end{pmatrix} = \det \begin{pmatrix} -\operatorname{Re} \mu'(0) & 0 \\ -\operatorname{Im} & -\frac{1}{2\pi} \end{pmatrix} = \frac{1}{2\pi} \operatorname{Re} \mu'(0) \neq 0$

Step 4: We have proved $\varphi(\lambda, T, r) = 0$ near $r = 0$ $T = 2\pi$, $\lambda = 0$, iff $\lambda = \lambda(r)$; $T = T(r)$

$\lambda = \lambda(r)$, $T = T(r)$

$x(s,t) \underset{s_i = r}{=} r e^{it} y_r + V(\lambda(r), T(r), r e^{it} y_r)$

derivative w.r. to r is zero, at $r = 0$

of minimal period 2π $\Longleftrightarrow$ $\hat{x}(s,t) := x(s, \frac{2\pi}{T} t)$

with min. period $T(s) = T(r)$ parameter

$\lambda(s) = \lambda(r)$ $\square$

Remarks:

1. If $\nu = \pm i$ are the only eigenvalues of $A(0)$ on Im-axis, we can also invoke (2d) center-mf, and normal form of semi-simple (actually: alg. simple) pair $\pm i$ induces $SO(2)$-action by $\Gamma = \{ e^{it} \mid t \in \mathbb{R}/2\pi\mathbb{Z} \}$, up to any finite order, ONLY, Funct. Ana. circumwents this difficulty: exact $SO(2)$ equivariance

2. There may be lots of other solutions, also perhaps periodic, near $\lambda = 0$, $x = 0$. Unless they posses period near 2π, they will not be discovered by L-S.

3. Large literature on case of resonances $\pm n i$, "mode interactions"

4. For PDE Hopf see Crandall & Rabinowitz